A Veeeeery Short Introduction to the Mathematical Theory of Non-Equilibrium Quantum Statistical Mechanics

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Content

0. Introduction

non-equilibrium thermodynamics as macroscopic field theory has its roots in phenomenological laws, as e.g.,

- heat conduction (Fourier, 1822)
- electric conduction (Ohm, 1826)

Paradigm

open system, i.e., “finite” sample $\mathcal{S}$ coupled to reservoirs $\mathcal{R}_r$

rigorous approach to:

non-equlibrium steady states? entropy production? Onsager relations? Kubo formula? Büttiker-Landauer formula? Fourier law? etc....??

use framework of C^*-algebraic quantum statistical mechanics...

1. Basic concepts of C^*-algebraic quantum statistical mechanics [JP02], [BR], [AJPP05],...

1.1 C^*-dynamical systems $(\mathcal{O}, \tau)$

- (unital) C^*-algebra $\mathcal{O}$

Banach $*$-algebra (algebra, involution $*$, norm $\|\cdot\|$, complete, $\|A^*\| = \|A\|$) with $\|A^*A\| = \|A\|^2$

- strongly continuous group $\mathbb{R} \ni t \mapsto \tau^t$ of $*$-automorphisms of $\mathcal{O}$

Examples $\mathcal{L}(\mathcal{H})$ (unital); $\mathcal{L}^\infty(\mathcal{H})$ (not unital)

1.2 States ω

- normalized ($\omega(1) = 1$), positive ($\omega(A^*A) \geq 0$) linear functional on $\mathcal{O}$
- $\mathcal{E}(\mathcal{O})$ set of states convex weak*-compact subset of $\mathcal{O}^*$; $U_{A_j,\epsilon}(\omega) = \{\omega' \colon |\omega'(A_j) - \omega(A_j)| < \epsilon\}$
- (τ, β)-KMS states. $(\mathcal{O}, \tau)$ C^*-dynamical system, $\beta \in \mathbb{R}$.

$$\omega(A\tau^{i\beta}(B)) = \omega(BA)$$

A, B in (subalgebra of) dense $*$-subalgebra $\mathcal{O}_\tau$ of $\mathcal{O}$ of entire analytic elements for τ

interpretation: systems in thermal equilibrium at temperature $1/\beta$

Example (i.g. formal) Gibbs state $\omega(A) = \mathrm{tr}(e^{-\beta H}A)/Z$ (e.g. finite system, Fermi: unique)

1.3 GNS representation

$\omega \in \mathcal{E}(\mathcal{O})$. [GNS] (unique) cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ of $\mathcal{O}$:

$$\omega(A) = (\Omega_\omega, \pi_\omega(A)\Omega_\omega)$$

- $\eta \in \mathcal{E}(\mathcal{O})$ ω-normal :⇔ exists density matrix ρ: $\eta(A) = \mathrm{tr}(\rho\, \pi_\omega(A))$

$\mathcal{N}_\omega$ set of all ω-normal states

1.4 (Concrete) von Neumann algebra

- commutant $\mathcal{H}$ Hilbert space, $\mathcal{M} \subseteq \mathcal{L}(\mathcal{H})$

$$\mathcal{M}' := \{A \in \mathcal{L}(\mathcal{H}) \,|\, [A, M] = 0,\ M \in \mathcal{M}\}$$

$\mathcal{M} \subseteq \mathcal{M}'' = \mathcal{M}^{(iv)} = \mathcal{M}^{(vi)} = \ldots;\quad \mathcal{M}' = \mathcal{M}''' = \mathcal{M}^{(v)} = \mathcal{M}^{(vii)} = \ldots$

- von Neumann algebra $\mathcal{M}'' = \mathcal{M}$

Examples $\mathcal{L}(\mathcal{H})$; not $\mathcal{L}^\infty(\mathcal{H})$ $(\mathcal{L}^\infty(\mathcal{H})' = \mathbb{C}1)$

$\mathcal{M}$ von Neumann algebra over $\mathcal{H}$, $\Omega \in \mathcal{H}$, $\mathcal{M}\Omega := \{A\Omega \,|\, A \in \mathcal{M}\}$

- $\Omega \in \mathcal{H}$ cyclic :⇔ $\overline{\mathcal{M}\Omega} = \mathcal{H}$
- $\Omega \in \mathcal{H}$ separating :⇔ $\Omega \in \ker A$: $A = 0$

1.5 Tomita-Takesaki theory

- $\mathcal{M}$ von Neumann algebra over $\mathcal{H}$, $\Omega \in \mathcal{H}$ cyclic and separating

- transfer $*$-involution on $\mathcal{M}$ to dense subspace $\mathcal{M}\Omega$ of $\mathcal{H}$:

$\theta : \mathcal{M} \to \mathcal{M}\Omega$, $A \mapsto A\Omega$ $\quad$ θ injective (separating), $\mathcal{M}\Omega$ dense (cyclic)

$$S_0 : \mathcal{M}\Omega \to \mathcal{M}\Omega, \quad S_0 A\Omega = A^*\Omega \qquad \begin{array}{ccc} \mathcal{M} & \stackrel{\theta^{-1}}{\longleftarrow} & \mathcal{M}\Omega \\ * \downarrow & & \downarrow S_0 \\ \mathcal{M} & \stackrel{\theta}{\longrightarrow} & \mathcal{M}\Omega \end{array}$$

- polar decomposition of $S = \overline{S}_0$, $S = J\sqrt{\Delta_\omega}$

modular conjugation J, modular operator Δ_ω

- [TT] $J\mathcal{M}J = \mathcal{M}'$, $\Delta_\omega^{it}\mathcal{M}\Delta_\omega^{-it} = \mathcal{M}$, $t \in \mathbb{R}$

- here: $\mathcal{M} \equiv \mathcal{M}_\omega := \pi_\omega(\mathcal{O})'' \subseteq \mathcal{L}(\mathcal{H}_\omega)$.

- $\omega \in \mathcal{E}(\mathcal{O})$ modular $:\Leftrightarrow$ Ω_ω is separating for $\mathcal{M}_\omega$

Example KMS state

1.6 Liouvilleans

$\omega \in \mathcal{E}(\mathcal{O})$ modular.

- natural cone $\mathcal{P} := \overline{\{AJA\Omega_\omega \,|\, A \in \mathcal{M}_\omega\}}$

$\eta \in \mathcal{N}_\omega$. exists unique $\Omega_\eta \in \mathcal{P}$: $\eta(A) = (\Omega_\eta, \pi_\omega(A)\Omega_\eta)$

- standard Liouvillean L

$(\mathcal{O}, \tau)$ C^*-dynamical system. exists unique self-adjoint L on $\mathcal{H}_\omega$:

$$\pi_\omega(\tau^t(A)) = e^{itL}\pi_\omega(A)e^{-itL}, \quad e^{-itL}\mathcal{P} \subseteq \mathcal{P}$$

1.7 Quantum statistical mechanics and modular theory

- study of ω-normal τ-invariant states reduces to the study of $\ker L$

$\eta \in \mathcal{N}_\omega$. η τ-invariant $\Leftrightarrow$ $L\Omega_\eta = 0$

- $\Delta_\omega = e^{\mathcal{L}_\omega}$. [T] ω is (τ, β)-KMS $\Leftrightarrow$ $\mathcal{L}_\omega = -\beta L$
- quantum Koopmanism: spectral properties of standard Liouvillean encode ergodic properties

Example [JP96] RTE if L has purely absolutely continous spectrum except for simple eigenvalue 0

1.8 Local perturbations

• $(\mathcal{O}, \tau)$ C^*-dynamical system, local perturbation $V = V^* \in \mathcal{O}$,
δ generator of τ^t δ $*$-derivation of $\mathcal{O}$: $\delta(A^*) = \delta(A)^*$, $\delta(AB) = \delta(A)B + A\delta(B)$, $A, B \in \mathcal{D}(\delta)$

• generator δ_V of perturbed dynamics $\tau_V^t := e^{t\delta_V}$

$$\delta_V(A) := \delta(A) + i[V, A]$$

• Dyson series

$$\tau_V(A) = \tau(A) + \sum_{n\geq 1} i^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \, [\tau^{t_n}(V), [\dots[\tau^{t_1}(V), \tau^t(A)]\dots]]$$

• $(\mathcal{O}, \tau_V)$ is C^*-dynamical system
• ω modular. standard Liouvillean for perturbed system L_V

$$L_V = L + V - JVJ$$

L standard Liouvillean for τ

1.9 Examples

1.9.1 Finite quantum systems

- C^*-dynamical systems $(\mathcal{O},\tau)$, $(\mathcal{O},\tau_V)$ $\mathcal{H}=\mathbb{C}^N$, $\mathcal{O}=\mathcal{L}(\mathcal{H})$; $H=H^*$, $V=V^*$

$$\tau^t(A)=e^{itH}Ae^{-itH}, \quad \tau_V^t(A)=e^{it(H+V)}Ae^{-it(H+V)}$$

- State any $\omega\in\mathcal{E}(\mathcal{O})$: $\omega(A)=\mathrm{tr}(\rho A)$, ρ density matrix on $\mathcal{H}$

Example unique (τ,β)-KMS state, $\beta\in\mathbb{R}$: $\rho=e^{-\beta H}/\mathrm{tr}(e^{-\beta H})$

- GNS representation $\lambda_j\geq 0,\psi_j$ eigenvv of ρ, complex conjugation on $\mathcal{H}$

$$\mathcal{H}_\omega=\mathcal{H}\otimes\mathcal{H}, \quad \pi_\omega(A)=A\otimes\mathbf{1}, \quad \Omega_\omega=\sum\sqrt{\lambda_j}\,\psi_j\otimes\psi_j$$

- Modular structure $J(\psi\otimes\phi)=\phi\otimes\psi$, $\mathcal{L}_\omega=\log\Delta_\omega=\log\rho\otimes\mathbf{1}-\mathbf{1}\otimes\log\rho$

- Standard Liouvillean $L=H\otimes\mathbf{1}-\mathbf{1}\otimes H$

No interesting thermodynamics for isolated finite quantum systems...but couple them to thermal reservoirs!

1.9.2 Free Fermi gas

• C^*-dynamical system$(\mathcal{O},\tau)$ 1-Fermion: Hilbert space $\mathfrak{h}$, Hamiltonian h

Examples free non-relativistic spinless electron of mass m: $\mathfrak{h} = L^2(\mathbb{R}^3)$, $h = p^2/2m$; spinless lattice Fermion: $\mathfrak{h} = l^2(\mathbb{Z}^d)$, $h = -\Delta$

Fock space $\mathfrak{F}(\mathfrak{h})$, *bounded* annihilation, creation operators $a(f), a^*(f)$
$\mathcal{O} = \mathrm{CAR}(\mathfrak{h})$ generated by $a^\sharp(f), f \in \mathfrak{h}$

$$\tau^t(a^\sharp(f)) = a^\sharp(e^{ith} f), \quad \tau^t(A) = e^{itH} A e^{-itH}, \; H = d\Gamma(h)$$

• Quasi-free, gauge-invariant state $T^* = T \in \mathcal{L}(\mathfrak{h})$, $0 \leq T \leq \mathbf{1}$

$$\omega(a^*(f_1)...a^*(f_n)a(g_1)...a(g_m)) = \delta_{m,n} \det\{(g_i, T f_j)\}$$

completely determined by 2-point function:

$$\omega(a^*(f)a(g)) = (g, Tf)$$

Examples $T = F(h)$: Fermi gas with energy density $F(E)$, e.g., $T = (1 + e^{\beta h})^{-1}$: unique (τ,β)-KMS state; cf. XY !; Pfaffian for *self-dual* CAR, cf. XY

- GNS representation [AW63] N number operator, Ω Fock vacuum

$$\begin{aligned} \mathcal{H}_\omega &= \mathfrak{F}(\mathfrak{h}) \otimes \mathfrak{F}(\mathfrak{h}), \quad \Omega_\omega = \Omega \otimes \Omega, \\ \pi_\omega(a(f)) &= a((\mathbf{1} - T)^{1/2} f) \otimes \mathbf{1} + (-1)^N \otimes a^*(T^{1/2} f) \end{aligned}$$

- Modular structure

$$\begin{aligned} J(\psi \otimes \phi) &= U\phi \otimes U\psi, \ U = (-1)^{N(N-1)/2} \\ \mathcal{L}_\omega &= \log \Delta_\omega = d\Gamma(S) \otimes \mathbf{1} - \mathbf{1} \otimes d\Gamma(S), \ S = \log T(1 - T)^{-1} \end{aligned}$$

- Standard Liouvillean

$$L = d\Gamma(h) \otimes \mathbf{1} - \mathbf{1} \otimes d\Gamma(h)$$

1.9.3 Lattice spin systems

c.f. 6.

2. Non-equilibrium steady states (NESS)

$(\mathcal{O}, \tau)$ C^*-dynamical system, $\omega \in \mathcal{E}(\mathcal{O})$, V local perturbation

$$\Sigma_+(\omega) := \text{weak}^*\text{-lim pt} \left\{ \frac{1}{T} \int_0^T dt\ \omega \circ \tau_V^t,\ T > 0 \right\}$$

• non-empty, weak*-compact subset of the weak*-compact set of states $\mathcal{E}(\mathcal{O})$ ($\mathcal{O}$ unital) containing τ_V-invariant NESS [R00]

• Abelian averaging: $\epsilon \int_0^\infty dt\, e^{-\epsilon t} \omega \circ \tau_V^t$, $\epsilon \downarrow 0$ (spectral deformation)

• $\eta \in \mathcal{N}_\omega$ (ω factor, weak asymptotic abelianness in mean).
[AJPP04] $\Sigma_+(\eta) = \Sigma_+(\omega)$

• structural properties of NESS, spectral characterization...

Example ω modular, ker L_V contains separating vector for $\mathcal{M}_\omega$: $\Sigma_+(\omega) \subseteq \mathcal{N}_\omega$

The response of the system to a local perturbation depends strongly on the nature of the initial state ω:

System near equilibirum: ω (τ, β)-KMS

$$\lim_{t\to\infty} \eta(\tau_V^t(A)) = \omega_V(A)$$

$\eta \in \mathcal{N}_\omega$, ω_V (τ_V, β)-KMS

- ergodic problem reduces to spectral analysis of Liouvillean L_V
- conceptually clear, spectral analysis done for few systems only

System far from equilibirum: η *not* normal w.r.t. some KMS state

- conceptual framework not well understood, the following two approaches are used (rigorous literature!)

3. The scattering approach to NESS

Møller morphism γ_+

$(\mathcal{O}, \tau)$ C^*-dynamical system, V local perturbation

$$\gamma_+ = \lim_{t \to \infty} \tau^{-t} \tau_V^t$$

algebraic analog of Hilbert space wave operator

NESS ω τ-invariant. $\omega_+ = \omega \circ \gamma_+$

Example ω (τ, β)-KMS $\Rightarrow \omega_+$ (τ_V, β)-KMS

- algebraic Cook criterion for the existence of γ_+ A in dense subset $\mathcal{O}_0$ of $\mathcal{O}$

$$\int_0^\infty dt \, \|[V, \tau_V^t(A)]\| < \infty$$

Remark difficult to verify in physically interesting models

Examples reduction to Hilbert space scattering problem for quasi-free systems [AP03], [AJPP ip]; locally perturbed Fermi gas [BM83]

4. The spectral approach to NESS [JP02]

ker L_V provides information about ω-normal, τ_V-invariant states; but thermodynamically interesting NESS *not* in $\mathcal{N}_\omega$!

usual approach: scattering theory

C-Liouvillean L^*

$(\mathcal{O}, \tau)$ C^*-dynamical system, ω modular, τ-invariant,

V local perturbation   assumptions about analytic continuation of $\Delta_\omega^{it} V \Delta_\omega^{-it}$, etc.

$$\mathsf{L}^* = L + V - J\Delta^{-1/2} V \Delta^{1/2} J$$

implements perturbed time evolution $\tau_V^t(A) = e^{it\mathsf{L}^*} A e^{-it\mathsf{L}^*}$, $A \in \mathcal{M}_\omega$

(Abelian) NESS are weak* limit points of $\epsilon/i\, \omega_{i\epsilon}$ for $\epsilon \downarrow 0$, where:

$$\omega_z(A) = i \int_0^\infty dt\; e^{izt} \omega(\tau_V^t(A)) = (\Omega_\omega, A(\mathsf{L}^* - z)^{-1} \Omega_\omega)$$

$\Rightarrow$ NESS described by resonance of L^* !

5. Entropy production

Phenomenology: entropy production σ is source term in local entropy density balance equation

$$\partial_t s + \mathrm{div}\,\mathrm{s} = \sigma$$

s entropy, s entropy flow; local formulation of 2$^{\mathrm{nd}}$ law of thermodynamics

system $\mathcal{S}$ coupled to thermal reservoirs $\mathcal{R}_k$ at temperatures $1/\beta_k$

$\Rightarrow$ stationary state: total entropy production in $\mathcal{S}$ equals entropy flux leaving $\mathcal{S}$: $-\sum_k \beta_r \phi_k$, ϕ_k energy current leaving $\mathcal{R}_k$

Ep(ω_+)

- $(\mathcal{O},\tau)$ C^*-dynamical system, ω τ-invariant, V local perturbation.

(A) exists C^* dynamics σ_ω: ω is $(\sigma_\omega,-1)$-KMS

Example $\omega = \otimes_k \omega_k$, ω_k is (τ_k,β_k)-KMS, satisfies (A) for $\sigma_\omega^t = \otimes_k \tau_k^{-\beta_k t}$, $\delta_\omega = -\sum_k \beta_k \delta_k$

Entropy production of locally perturbed system $(\mathcal{O}, \tau_V)$ in NESS $\omega_+ \in \Sigma_+(\omega)$:

$$\text{Ep}(\omega_+) := \omega_+(\delta_\omega(V))$$

Example $\omega = \otimes_k \omega_k$, ω_k is (τ_k, β_k)-KMS: $\text{Ep}(\omega_+) = -\sum_k \beta_k \omega_+(\delta_k(V)) = -\sum_k \beta_k \phi_k$

[JP02]:

- entropy production as asymptotic rate of decrease of relative entropy (cf. [S78], [LS78]; [OHI88], [O89], [O91])

$\text{Ent}(\eta \circ \tau^t | \omega) - \text{Ent}(\eta | \omega) = \int_0^t ds\, \eta \circ \tau^s(\delta_\omega(V))$

- $\text{Ep}(\omega_+) \geq 0$
- ω_+ ω-normal $\Rightarrow$ $\text{Ep}(\omega_+) = 0$
- ω_+ weakly ergodic. $\text{Ep}(\omega_+) = 0 \Rightarrow \omega_+$ ω-normal
- ω_+ KMS, iff $\text{Ep} = 0$ for sufficiently many local perturbations

6. Application of the scattering approach: XY model

• "integrable" models as essential tools in development of equilibrium statistical mechanics - *out* of equilibrium: dynamics crucial!

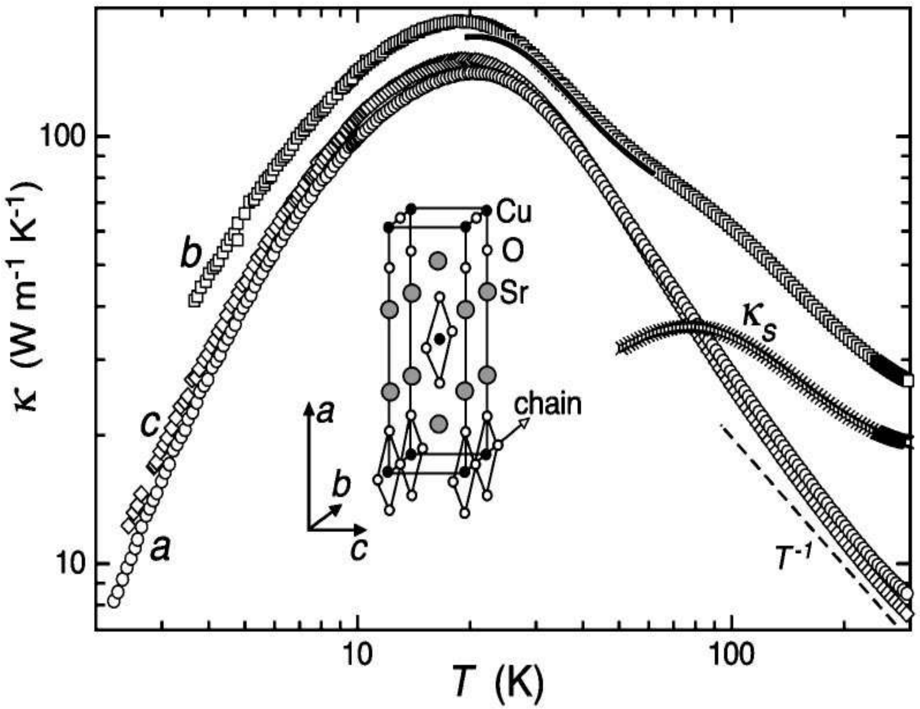
• XY model one of few systems for which explicit knowledge of dynamics available   Jordan-Wigner transformation!

• integrability may be traced back to infinite family of charges

master symmetries [BF85], [A90]

integrability relates to anomalous transport:
theoretically: overlap of current with charges prevents current-current correlation to decay to zero: ideal thermal conductivity
numerically: Fourier law violated for "integrable" systems
experimentally: anomalous transport properties in low-dimesional magnetic systems, e.g. Heisenberg models

• Sr_2CuO_3



• best physical realization of $1d, S = 1/2$ XYZ Heisenberg model: interchain/intrachain interaction: $\sim 10^{-5}$ ($PrCl_3$: XY)

• anomalously enhanced conductivity along chain direction [S00]

electric insulator; T high: spinons $\gg$ phonons, limited by defects & phonons

XY chain

inifinite chain of spins interacting anisotropically with two nearest neighbors and with external magnetic field: $\gamma \in (-1, 1)$, $\lambda \in \mathbb{R}$

$$H = -\frac{1}{4} \sum_{x \in \mathbb{Z}} \left((1 + \gamma)\sigma_1^{(x)} \sigma_1^{(x+1)} + (1 - \gamma)\sigma_2^{(x)} \sigma_2^{(x+1)} + 2\lambda \sigma_3^{(x)} \right)$$

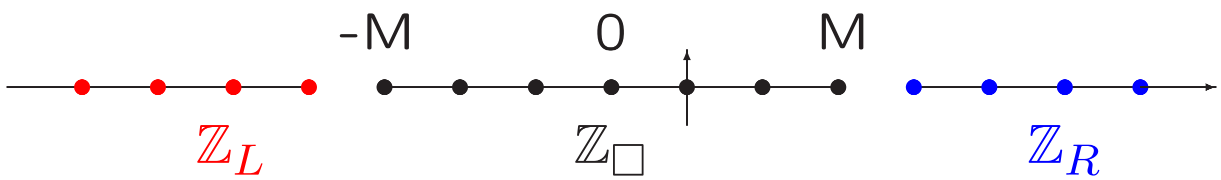
6.1 Non-equilibrium setting [AP03]

remove bonds $(-M-1, -M)$ and $(M, M+1)$

$\Rightarrow$ 3 decoupled subsystems with (τ_L, β_L), $(\tau_\square, 0)$, (τ_R, β_R)-KMS states

$$\omega_0 = \omega_L^{\beta_L} \otimes \omega_\square \otimes \omega_R^{\beta_R}$$

infinite half-chains $\mathbb{Z}_L$, $\mathbb{Z}_R$ play role of thermal reservoirs to which finite subsystem $\mathbb{Z}_\square$ is attached via coupling $V = H - H_0$



6.2 Kinematics

• algebra of observables $\mathcal{O} \equiv \mathfrak{S}$

uniformly hyperfinite quasi-local C^* algebra $\mathfrak{S}$ over $\mathbb{Z}$, i.e., associate Hilbert space $\mathcal{H}_{\{x\}} = \mathbb{C}^2$ to $x \in \mathbb{Z}$, and for *finite* subset Λ of $\mathbb{Z}$,

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_{\{x\}}, \quad \mathfrak{S}_\Lambda = \mathcal{L}(\mathcal{H}_\Lambda)$$

infinite tensor product of $\mathcal{L}(\mathcal{H}_{\{x\}})$ for x in *arbitrary* subset $\mathcal{Z}$ of $\mathbb{Z}$,

$$\mathfrak{S}_\mathcal{Z} = \bigcup_{\Lambda \subset \mathcal{Z}} \mathfrak{S}_\Lambda,$$

i.e., uniform limit of polynomials in Pauli matrices $\sigma_\alpha^{(x)}$, $\alpha = 0, 1, 2, 3$

$$\mathfrak{S} = \mathfrak{S}_\mathbb{Z}, \quad \mathfrak{S}_L = \mathfrak{S}_{\{x<-M\}}, \quad \mathfrak{S}_\square = \mathfrak{S}_{\{-M \le x \le M\}}, \quad \mathfrak{S}_R = \mathfrak{S}_{\{x>M\}}$$

6.3 Dynamics

- local XY Hamiltonian $H_\Lambda = \sum_{X \subseteq \Lambda} \phi(X)$, interaction $\phi : X \to \mathfrak{S}_X$

$$\phi(X) = \begin{cases} -\frac{1}{2}\lambda\sigma_3^{(x)}, & X = \{x\}, \\ -\frac{1}{4}\{(1+\gamma)\sigma_1^{(x)}\sigma_1^{(x+1)} + (1-\gamma)\sigma_2^{(x)}\sigma_2^{(x+1)}\}, & X = \{x, x+1\}, \\ 0, & \text{otherwise} \end{cases}$$

short range, two-body

- local perturbed dynamics, its thermodynamic limit

$$\tau_\Lambda^t(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}, \quad \tau^t(A) = \lim_{\Lambda \uparrow \mathbb{Z}} \tau_\Lambda^t(A)$$

$\Rightarrow$ perturbed C^*-dynamical system $(\mathfrak{S}, \tau)$

- free dynamics from local perturbation V

$$V = \phi(\{-M-1, -M\}) + \phi(\{M, M+1\})$$

$\Rightarrow$ free C^*-dynamical system $(\mathfrak{S}, \tau_0)$

$$\mathfrak{S} = \mathfrak{S}_L \otimes \mathfrak{S}_\square \otimes \mathfrak{S}_R, \quad \tau_0^t = \tau_L^t \otimes \tau_\square^t \otimes \tau_R^t$$

6.4 Jordan-Wigner transformation: key to "exact solution"

$$a_x := TS^{(x)}(\sigma_1^{(x)} - i\sigma_2^{(x)})/2, \quad S^{(x)} = \begin{cases} \sigma_3^{(1)} \cdots \sigma_3^{(x-1)}, & x > 1 \\ 1, & x = 1 \\ \sigma_3^{(x)} \cdots \sigma_3^{(0)}, & x < 1 \end{cases}$$

a_x, a_x^* generate CAR algebra! T for *two*-sided chain ($\mathfrak{F} \otimes_{\theta_-} \mathbb{Z}_2$ crossed product) [A84]

- interaction becomes quadratic

$$\phi(X) = \begin{cases} -\frac{1}{2}\lambda(2a_x^* a_x - 1), & X = \{x\} \\ \frac{1}{2}\{a_x^* a_{x+1} + a_{x+1}^* a_x + \gamma(a_x^* a_{x+1}^* + a_{x+1} a_x)\}, & X = \{x, x+1\} \\ 0, & \text{otherwise} \end{cases}$$

- dynamics become Bogoliubov automorphisms

$$\tau^t(B(f)) = B(\mathrm{e}^{ith} f), \quad \tau_0^t(B(f)) = B(\mathrm{e}^{ith_0} f)$$

self-dual CAR algebra with $B(f) = \sum_{x \in \mathbb{Z}} f_+(x)\, a_x^* + f_-(x)\, a_x, \; f = (f_+, f_-) \in \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ [A71]
with 1-particle Hamiltonians

$$h = (\cos\xi - \lambda) \otimes \sigma_3 - \gamma \sin\xi \otimes \sigma_2, \quad h_0 = h - v = h_L \oplus h_\square \oplus h_R$$

Fourier variable ξ , V (self-dual) 2nd quantization of v

Non-equilibrium properties

6.5 Existence and uniqueness of NESS

Theorem

Let $\beta_L, \beta_R \in \mathbb{R}$, $M \in \mathbb{N}$. Then:

$$\Sigma_+(\omega_0) \;=\; \{\omega_+\}$$

Proof

- [A84] $\beta_L = \beta_R \equiv \beta$: ω_+ unique (τ, β)-KMS, RTE
- [KB] $1_{ac}(h) = 1$, $v \in \mathcal{L}^0$: $w_-^* = \text{s}-\lim \text{e}^{ith_0}\,\text{e}^{-ith}$ exists, complete
- $\|B(f)\| \leq \|f\|$, norm convergence

$$\tau_0^{-t}\tau^t(B(f)) = B(e^{-ith_0}e^{ith}f) \;\Rightarrow\; B(w_-^* f) = \gamma_+(B(f)) \;\Rightarrow\; \omega_+ = \omega_0 \circ \gamma_+$$

□

6.6 2-point operator T_+ of ω_+

Theorem

ω_+ has 2-point function $\omega_+(B^*(f)B(g)) = (f, T_+ g)$

$$T_+ = (1 + e^{k_+})^{-1}, \quad k_+ = (\beta + \delta \, \text{sign} \, v_-)\, h$$

v_- asymptotic velocity (in past), $\beta = (\beta_R + \beta_L)/2$, $\delta = (\beta_R - \beta_L)/2$

Proof

- $\omega_+ = \omega_0 \circ \gamma_+ \Rightarrow T_+ = w_- T_0 w_-^*$
- partial wave operators w_α, asymptotic projections P_α

$j_\alpha : \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2 \to \ell^2(\mathbb{Z}_\alpha) \otimes \mathbb{C}^2$, $\alpha = L, R$: canonical projections

$$w_\alpha^* = \underset{t \to -\infty}{\text{s}-\lim} \, \mathrm{e}^{ith_\alpha} \, j_\alpha \, \mathrm{e}^{-ith}, \quad P_\alpha = \underset{t \to -\infty}{\text{s}-\lim} \, \mathrm{e}^{ith} \, j_\alpha^* j_\alpha \, \mathrm{e}^{-ith}$$

[KB], [DS] existence,completeness of P_α, $w_-^* = \sum_{\alpha \in \{L,R\}} j_\alpha^* \, w_\alpha^*$,
$h_\alpha \, w_\alpha^* = w_\alpha^* \, h$, $P_\alpha = w_\alpha w_\alpha^*$, $P_L + P_R = I$, $[P_\alpha, h] = 0$

• from $T_0 = (1 + e^{k_0})^{-1}$ with $k_0 = \beta_L\, h_L \oplus 0 \oplus \beta_R\, h_R$

$$T_+ = (1 + e^{k_+})^{-1}, \quad k_+ = \beta\, h + \delta\, (P_R - P_L)\, h$$

• since $1_{ac}(h) = 1$ $\quad (x = -i\partial_\xi \otimes \mathbf{1},\ x_t = e^{-ith} x e^{ith})$

$$P_R - P_L = \text{s}-\lim_{t\to\infty} \text{sign}\, \frac{x_t}{t}$$

• solve $\dot{x}_t = p_t$: $\text{s}-\lim_{t\to\infty} \frac{x_t}{t} = \mu h$ $\quad (\mu = \mathbf{p}\mathbf{h}/\mathbf{h}^2$ with $p = -i[h, x] = \mathbf{p}\cdot\sigma,\ h = \mathbf{h}\cdot\sigma)$

$$v_- := \text{s-res-}\lim_{t\to\infty} \frac{x_t}{t} = \mu h, \quad P_R - P_L = \text{sign}\, v_-$$

□

Remarks

• since $k_+ = \beta_L h P_L \oplus \beta_R h P_R$ NESS ω_+ describes mixture of two *independent* species: left-movers from ran P_R carry β_R, right-movers from ran P_L carry β_L (cf. [ACF98])

• further properties: ω_+ is attractive, independent of M, translation invariant, primary, modular, quasi-free, KMS iff $\beta_L = \beta_R$, singular w.r.t. ω_0,...

6.7 Entropy production

entropy production in $\omega_+ \in \Sigma_+(\omega_0)$

$$\mathrm{Ep}(\omega_+) = \beta_L \omega_+(\Phi_L) + \beta_R \omega_+(\Phi_R)$$

$\Phi_L = -i[H, H_L]$, $\Phi_R = -i[H, H_R]$: Heat fluxes $\mathbb{Z}_L$, $\mathbb{Z}_R \to \mathbb{Z}_\square$

Theorem

$$\begin{aligned} \mathrm{Ep}(\omega_+) &= \frac{\delta}{4}\int_0^{2\pi} \frac{d\xi}{2\pi} |\mathbf{p}\cdot\mathbf{h}| \frac{\mathrm{sh}\,\delta|h|}{\mathrm{ch}^2(\beta|h|/2) + \mathrm{sh}^2(\delta|h|/2)} \\ \mathrm{Ep}(\omega_+) &> 0 \qquad \text{if } \beta_L \neq \beta_R \end{aligned}$$

Proof explicit computation! □

Remark

[AJPP ip] non-equilibrium properties for general quasi-free systems

7. Application of the spectral approach: Spin-Fermion model

- finite quantum system $\mathcal{S}$ (spin): $(\mathcal{O}_\mathcal{S}, \tau_\mathcal{S})$ (cf. 1.9.1)
- reservoirs $\mathcal{R}_r$ $(r = L, R)$: $(\mathcal{O}_r, \tau_r)$ (cf. 1.9.2)
- $V_L = \varphi(\alpha_L) \otimes Q_L \otimes 1,\ V_R = 1 \otimes Q_R \otimes \varphi(\alpha_R),\ V = V_L + V_R$

Segal field operator φ quantizing (sufficiently regular) coupling functions α_r, and $Q_r \in \mathcal{L}(\mathcal{H}_\mathcal{S})$

- initial state $\omega = \omega_L \otimes \omega_\mathcal{S} \otimes \omega_R$ $\quad$ $\omega_\mathcal{S}$ trace state, ω_r (τ_r, β_r)-KMS

lowest order entropy production [JP02]

$$
\begin{aligned}
\mathrm{Ep}(\omega_+^\lambda) &= \lambda^2 \sigma(\rho_0) + \mathcal{O}(\lambda^3) \\
-\sigma(\rho_0) &= \beta_L\, \omega_{\rho_0}(K_L H_\mathcal{S}) + \beta_R\, \omega_{\rho_0}(K_R H_\mathcal{S})
\end{aligned}
$$

spectral theory of C-Liouvillean: resonances from complex translation; $\sigma(\rho_0)$ entropy production in van Hove weak coupling limit: K_L, K_R Davies generators, $\omega_{\rho_0}(\cdot) = \mathrm{tr}(\rho_0 \cdot)$, $\ker(K_L + K_R) = \{\rho_0\}$

strict positivity of entropy production [AS ip] $\quad$ small, good coupling [LS78]

$$
\{H_\mathcal{S}, Q_r\} = \mathbb{C}1 \implies \sigma(\rho_0) > 0 \implies \mathrm{Ep}(\omega_+^\lambda) > 0
$$

Examples $\mathcal{H}_\mathcal{S} = \mathbb{C}^2$, single spin; $\mathcal{H}_\mathcal{S} = \mathbb{C}^4$, XY

8. Outlook

- correlation functions: long range out of equilibrium ?
- integrability, conserved charges, and Fourier law ?
- Gallavotti-Cohen symmetry of entropy production ?
- non-equilibrium phase transitions ?
- etc....