On the decay of spin-spin correlations in quasi-free states

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Content

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1. Motivation

• unusual transport properties in low-dimensional magnetic systems: intensively studied experimentally and theoretically

best physical realisation of $1d, S = 1/2$ Heisenberg model: interchain/intrachain interaction small

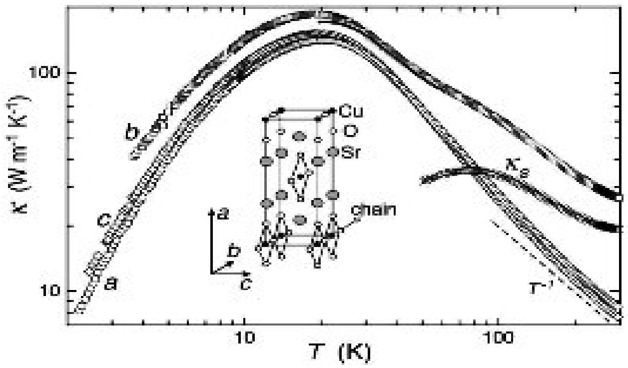


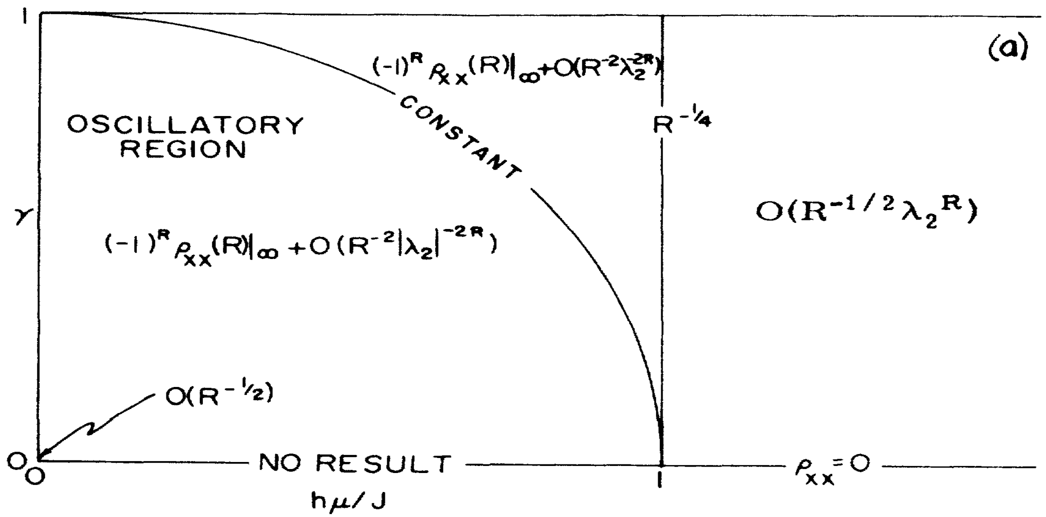
FIG. 2. Temperature dependence of the thermal conductivity of Sr_2CuO_3 along the a, b, and c axes. κ_s is the calculated spinon thermal conductivity along the b axis. The solid line is the estimated sum of spinon and phonon thermal conductivities assuming that the spinon mean free path is equal to the distance between bond defects (see text). The schematic crystal structure is shown in the inset.

[Sologubenko et al. 00] anomalously enhanced conductivity along chain direction

• XY chain one of simplest non-trivial models in quantum statistical mechanics: testing ground for new ideas

• spin-spin correlations in XY chain at $T = 0$ and $T > 0$ in equilibrium
[Lieb, Schultz, Mattis, Barouch, McCoy,...60']

$H_{XY} = -\frac{1}{4}\sum_{x\in\mathbb{Z}}\left((1+\gamma)\sigma_1^{(x)}\sigma_1^{(x+1)} + (1-\gamma)\sigma_2^{(x)}\sigma_2^{(x+1)} + 2\lambda\sigma_3^{(x)}\right)$



$h\mu/J = \lambda$

• general translation invariant quasi-free states:
simple criterion for exponential decay of spin-spin correlations ?

2. Setting

2.1 CAR algebra

- CAR algebra $\mathcal{A}$ over $\ell^2(\mathbb{Z})$

$$B(F) = a^*(f_1) + a(\bar{f}_2)$$

$F = [f_1, f_2] \in \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$

- antiunitary involution $J[f_1, f_2] = [\bar{f}_2, \bar{f}_1]$
- self-dual CAR

$$\{B^*(F), B(G)\} = (F, G), \quad B^*(F) = B(JF)$$

Araki's self-dual CAR algebra $\mathcal{A}_{sdc}(\mathfrak{h}, J) \simeq \mathcal{A}_{car}(P\mathfrak{h})$, basis projection $JPJ = 1 - P$

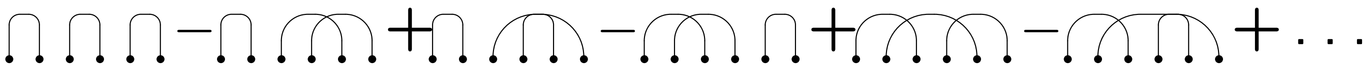
2.2 Quasi-free states

- ω vanishes on the odd polynomials, and

$$\omega(B(F_1)...B(F_{2n})) = \sum_{\pi} \mathrm{sign}(\pi) \prod_{j=1}^{n} \omega(B(F_{\pi(2j-1)})B(F_{\pi(2j)}))$$

$F_j \in \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ with $j = 1, ..., 2n$ and every $n \in \mathbb{N}$, sum over pairings of $\{1, 2, ..., 2n\}$

- correlation $C(n) = \omega(B(F_1)...B(F_{2n}))$

$n = 3$ 

- correlation matrix $\Omega(n) \in \mathbb{C}^{2n \times 2n}$

$$\Omega(n)_{kl} = \begin{cases} \omega(B(F_k)B(F_l)), & k < l \\ 0, & k = l \\ -\omega(B(F_l)B(F_k)), & k > l \end{cases} \quad \Rightarrow \quad C(n) = \mathrm{pf}\ \Omega(n)$$

pf A Pfaffian of A

$$\Omega(n)^t = -\Omega(n) \quad \Rightarrow \quad C(n)^2 = \det \Omega(n)$$

Assumption 1 translation invariance

$*$-automorphism $\tau_x(B(F)) = B(U_x F)$, $\quad U_x = u_x \oplus u_x$, $\quad (u_x f)(y) = f(y-x)$

$$\omega \circ \tau_x = \omega$$

Assumption 2 form factors F_j

$$F_{2j-1} = U_j G_0, \qquad F_{2j} = U_j G_1$$

Assumption 3 initial form factors G_0, G_1

$$G_0 = [-\delta_{-1}, \delta_{-1}], \qquad G_1 = [\delta_0, \delta_0]$$

spin-spin after Jordan-Wigner: $\sigma_1^{(0)}\sigma_1^{(n)} = B(F_1)B(F_2) \cdot \ldots \cdot B(F_{2n})$

$\Rightarrow$ $\Omega(n)$ 2×2 block Toeplitz matrix

$$\Omega(n)_{k+2,l+2} = \Omega(n)_{k,l}$$

$\Rightarrow$ analysis of Toeplitz operators

3. Toeplitz theory

$\ell^2_N = \{f : \mathbb{N} \to \mathbb{C}^N \,|\, \|f\| < \infty\}$, $\|f\| = \left(\sum_{i=1}^{\infty} \|f_i\|^2_{\mathbb{C}^N}\right)^{1/2}$

$L^\infty_{N\times N} = \{\phi : \mathbb{T} \to \mathbb{C}^{N\times N} \,|\, \phi_{ij} \in L^\infty(\mathbb{T}), i, j = 1, ..., N\}$

Toeplitz operator on ℓ^2_N, $f \mapsto \{\sum_{j=1}^{\infty} a_{i-j}\, f_j\}_{i=1}^{\infty}$, $a_x \in \mathbb{C}^{N\times N}$
bounded $\Leftrightarrow$ exists $a \in L^\infty_{N\times N}$ s.t.

$$a_x = \int_0^{2\pi} \frac{d\xi}{2\pi}\, a(\xi) e^{-ix\xi}$$

symbol a, and

$$T[a] = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

- finite section

$$T_n[a] = P_n T[a] P_n|_{\mathrm{Im}\, P_n}$$

$P_n(\{x_1, ..., x_n, x_{n+1}, ...\}) = \{x_1, ..., x_n, 0, 0, ...\}$

- norm of Toeplitz operators

$$\|T[a]\| = \|a\|_\infty,$$

$\|a\|_\infty = \operatorname{ess\,sup}_{t\in\mathbb{T}} \|a(t)\|_{\mathcal{L}(\mathbb{C}^N)}$

Avram-Parter

$t_1^{(n)}, ..., t_{Nn}^{(n)}$ singular values of $T_n[a]$, $g \in C_0(\mathbb{R})$

$$\lim_{n\to\infty} \frac{1}{Nn} \sum_{j=1}^{Nn} g((t_j^{(n)})^2) = \frac{1}{N} \int_0^{2\pi} \frac{d\xi}{2\pi} \operatorname{tr} g(a^*(e^{i\xi})a(e^{i\xi}))$$

4. Asymptotics

4.1 Symbol

2-point operator S

$$\omega(B^*(F)B(G)) = (F, SG), \quad 0 \leq S \leq 1, \ JSJ = 1 - S$$

$\omega_s \equiv \omega$

- translation invariance $\omega_S \circ \tau_x = \omega_S \Leftrightarrow [S, U_x] = 0$

- Fourier $s = \mathcal{F}S\mathcal{F}^*$ on $L^2(\mathbb{T}) \oplus L^2(\mathbb{T})$: multiplication by $s_{ij} \in L^\infty(\mathbb{T})$

$\mathcal{F} = \hat{} \oplus \hat{}$

Lemma If Assumptions 1,2,3 hold, and if $(\mathcal{F}[JG_\alpha], s\mathcal{F}[G_\beta])_{\mathbb{C}^2} \in L^\infty(\mathbb{T})$, then $\Omega = T[a] \in \mathcal{L}(\ell_2^2)$ with symbol

$$a = \begin{bmatrix} (\mathcal{F}[JG_0], s\mathcal{F}[G_0])_{\mathbb{C}^2} + 1 & (\mathcal{F}[JG_0], s\mathcal{F}[G_1])_{\mathbb{C}^2} \\ (\mathcal{F}[JG_1], s\mathcal{F}[G_0])_{\mathbb{C}^2} & (\mathcal{F}[JG_1], s\mathcal{F}[G_1])_{\mathbb{C}^2} - 1 \end{bmatrix},$$

and

$$\Omega(n) = T_n[a].$$

4.2 Exponential decay

$$\mathfrak{S} = \{\xi \in [0, 2\pi) \mid 1/2 \notin \operatorname{spec} s(e^{i\xi}) \not\subseteq \{0,1\}\}$$

Theorem If $\mathfrak{L}(\mathfrak{S}) > 0$, then

$$\limsup_{n\to\infty} \frac{1}{n} \log |C(n)| \le \frac{1}{2} \int_{\mathfrak{S}} \frac{d\xi}{2\pi} \log |\det(2s(e^{i\xi}) - 1)| < 0.$$

Remark generalisation to

$(JU_x G_\alpha, G_\beta) = 2(-1)^{\alpha+1}\delta_{\alpha\beta}\delta_{x0}$, $JG_\alpha = (-1)^{\alpha+1}G_\alpha$, $(\mathcal{F}[G_\alpha], \mathcal{F}[G_\beta])_{\mathbb{C}^2} = 2\delta_{\alpha\beta}$ and beyond

Proof

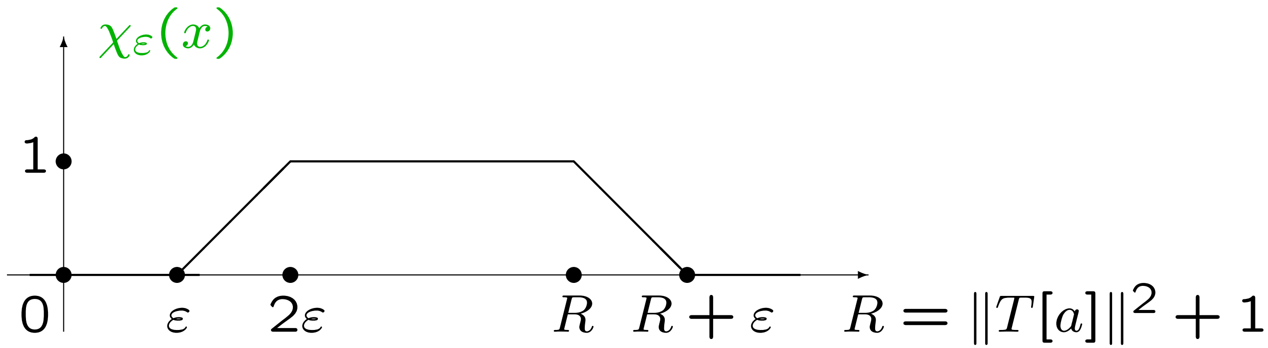
- since $|C(n)|^2 = |\det T_n[a]|$, estimate singular values of $T_n[a]$

$$0 \le t_1^{(n)} \le t_2^{(n)} \le \ldots \le t_{2n}^{(n)} = \|T_n[a]\|$$

- $N_{\varepsilon,n} = \sharp\{t_j^{(n)} \,|\, (t_j^{(n)})^2 \le \varepsilon\}$ with $\varepsilon > 0$

$$\Rightarrow \quad |C(n)|^4 = \prod_{j=1}^{2n} (t_j^{(n)})^2 \le \varepsilon^{N_{\varepsilon,n}} \prod_{j=N_{\varepsilon,n}+1}^{2n} (t_j^{(n)})^2$$

- define



$$g_\varepsilon(x) = \begin{cases} 0, & x \le 0 \\ \chi_\varepsilon(x) \log x, & x > 0 \end{cases}$$

$$\Rightarrow \quad \log |C(n)|^4 \le \sum_{j=N_{\varepsilon,n}+1}^{2n} \log((t_j^{(n)})^2) \le \sum_{j=1}^{2n} g_\varepsilon((t_j^{(n)})^2)$$

$$[\mathrm{AP}] \Rightarrow \limsup_{n\to\infty} \frac{1}{n} \log |C(n)| \le \frac{1}{4}\int_0^{2\pi} \frac{d\xi}{2\pi} \,\mathrm{tr}\, g_\varepsilon(a^*(e^{i\xi})a(e^{i\xi}))$$

- to estimate trace term, define

$$\begin{aligned} \mathfrak{S}_c &= \{\xi \in [0,2\pi) \mid 1/2 \notin \mathrm{spec}\, s(e^{i\xi})\}, \\ \mathfrak{S}_b &= \{\xi \in [0,2\pi) \mid \mathrm{spec}\, s(e^{i\xi}) \not\subseteq \{0,1\}\}, \\ \mathfrak{R}_\delta &= \{\xi \in [0,2\pi) \mid \mathrm{spec}\, a^*(e^{i\xi})a(e^{i\xi}) \subseteq (\delta, \|a\|_\infty^2]\}, \end{aligned}$$

- $\det a(e^{i\xi}) = -\det\,[2s(e^{i\xi}) - 1]$

- $\mathfrak{S}_c = \{\xi \in [0,2\pi) \mid \mathrm{spec}\, a^*(e^{i\xi})a(e^{i\xi}) \subseteq (0, \|a\|_\infty^2]\}$
- $\mathfrak{L}(\mathfrak{R}_\delta) > 0$ for sufficiently small δ since $\mathfrak{L}(\mathfrak{S}_c) \ge \mathfrak{L}(\mathfrak{S}) > 0$

$$\Rightarrow \quad \tfrac{1}{4}\int_0^{2\pi} \tfrac{d\xi}{2\pi}\,\mathrm{tr}\, g_\varepsilon(a^*(e^{i\xi})a(e^{i\xi})) \le \tfrac{1}{2}\int_{\mathfrak{R}_{2\varepsilon}} \tfrac{d\xi}{2\pi} \log|\det a(e^{i\xi})|$$

$\delta = 2\varepsilon$ small enough

- $-1 \leq 2s(e^{i\xi}) - 1 \leq 1 \Rightarrow |\det a(e^{i\xi})| \leq 1$
- $\mathfrak{S}_b = \{\xi \in [0, 2\pi) \mid |\det a(e^{i\xi})| \neq 1\}$
- $\mathfrak{L}(\mathfrak{S}_b \cap \mathfrak{R}_{2\varepsilon}) > 0$

$$\Rightarrow \quad \frac{1}{2}\int_{\mathfrak{R}_{2\varepsilon}} \frac{d\xi}{2\pi} \log |\det a(e^{i\xi})| = \frac{1}{2}\int_{\mathfrak{S}_b \cap \mathfrak{R}_{2\varepsilon}} \frac{d\xi}{2\pi} \log |\det a(e^{i\xi})| < 0$$

$\square$

5. Examples

5.1 Decay out of equilibrium

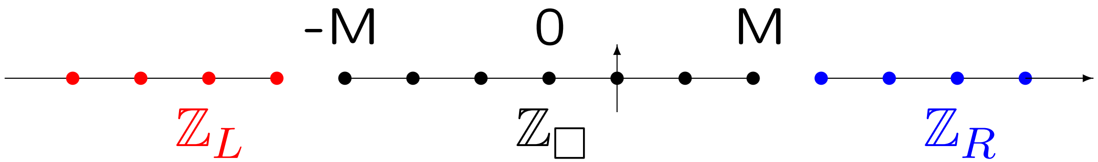
• non-equilibrium steady state ω of the XY chain [A,Pillet 03]

$$H = -\frac{1}{4}\sum_{x\in\mathbb{Z}}\Big((1+\gamma)\sigma_1^{(x)}\sigma_1^{(x+1)} + (1-\gamma)\sigma_2^{(x)}\sigma_2^{(x+1)} + 2\lambda\sigma_3^{(x)}\Big)$$

γ anisotropy, λ magnetic field

• quasi-free initial state

$\omega_0 = \omega_L^{\beta_L} \otimes \omega_\square \otimes \omega_R^{\beta_R}$



• scattering on 1-particle space $\omega = \omega_0 \circ \gamma$ with $\gamma = \lim_{t\to\infty} \tau_0^{-t}\tau^t$

τ_0 decoupled dynamics, τ dynamics with closed bonds

$$s(e^{i\xi}) = \left(1 + e^{\beta\, h(e^{i\xi}) + \delta\, k(e^{i\xi})}\right)^{-1}$$

$\beta = (\beta_R + \beta_L)/2$, $\delta = (\beta_R - \beta_L)/2$, $h(e^{i\xi}) = (\cos\xi - \lambda)\otimes\sigma_3 - \gamma\sin\xi\otimes\sigma_2$, $k(e^{i\xi}) = \mathrm{sign}(\kappa(e^{i\xi}))\,\mu(e^{i\xi})\otimes\sigma_0$

- spectrum of $s(e^{i\xi})$

$$\lambda_\pm(e^{i\xi}) = \frac{1}{2}(1+\varphi_{\delta,\beta}(e^{i\xi})\,\mathrm{sign}\,\kappa(e^{i\xi}) \pm \varphi_{\beta,\delta}(e^{i\xi}))$$

$$\varphi_{\alpha,\alpha'} = \frac{\mathrm{sh}\,(\alpha\mu)}{\mathrm{ch}\,(\alpha\mu)+\mathrm{ch}\,(\alpha'\mu)}, \qquad \mu = \left((\cos\xi-\lambda)^2+\gamma^2\sin^2\xi\right)^{1/2}$$

critical parameters $\mathcal{H} = \{(\gamma,\lambda) \in \{0\}\times(-1,1)\cup(-1,1)\times\{\pm 1\}\}$

- non-critical case $(\gamma,\lambda)\notin\mathcal{H}$: $\mathfrak{S}=[0,2\pi)$, and gap (strengthens [AB06])
- critical case $(\gamma,\lambda)\in\mathcal{H}$: $\mathfrak{S}=[0,2\pi)$, no gap

$$\limsup_{n\to\infty}\frac{1}{n}\log|C(n)| \le \frac{1}{2}\int_{\mathfrak{S}}\frac{d\xi}{2\pi}\log[\mathrm{th}(\beta_L\mu(e^{i\xi})/2)\,\mathrm{th}(\beta_R\mu(e^{i\xi})/2)] < 0$$

5.2 Decay in equilibrium at positive temperature

$$s(e^{i\xi}) = \left(1 + e^{\beta\, h(e^{i\xi})}\right)^{-1}$$

$\delta = (\beta_R - \beta_L)/2 = 0$

• change in decay rate ($\beta_L < \beta_R$)

$2 \log \mathrm{th}(\beta_L \mu/2) \leq \log[\mathrm{th}(\beta_L \mu/2)\mathrm{th}(\beta_R \mu/2)] \leq 2 \log \mathrm{th}(\beta_R \mu/2)$

$\Rightarrow$ faster decay out of equilibrium than in thermal equilibrium (at β_R)

examples with $\mathfrak{L}(\mathfrak{S}) = 0 \quad \Rightarrow$ Theorem not applicable
which behavior possible ?

5.3 Decay at vanishing temperature

$$s(e^{i\xi}) = \frac{1}{2}\left(1 + \frac{1}{\mu(e^{i\xi})}\begin{bmatrix} \cos\xi - \lambda & -i\gamma\sin\xi \\ i\gamma\sin\xi & -(\cos\xi - \lambda) \end{bmatrix}\right)$$

$\delta = 0$, $\beta = \infty$

- vanishing long range order, slow decay

$\gamma = \lambda = 0 \quad \Rightarrow \quad n^{-1/2}$

- nonvanishing long range order

$0 < \gamma, \lambda < 1 \quad \Rightarrow \quad \lim_{n\to\infty} |C(n)| = \kappa(\gamma, \lambda) > 0$

$\gamma^2 + \lambda^2 = 1 \quad \Rightarrow \quad |C(n)| = \kappa(\gamma, \lambda)$ for all n

- constantly vanishing correlation

$\gamma = 0,\ \lambda \geq 1 \quad \Rightarrow \quad C(n) = 0$

Remark construction of simple examples with $0 \leq s(e^{i\xi}) \leq 1$, $\sigma_1 \overline{s}(e^{-i\xi})\sigma_1 = 1 - s(e^{i\xi})$