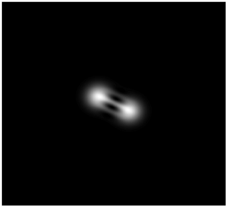
From Bose-Einstein Condensation to Monster Waves: The Nonlinear Hartree Equation, and some of its Large Coupling Phenomena



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Model Equations in Bose-Einstein Condensation and Related Topics
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Contents

1. Motivation: The Hartree equation is ubiquitous

Definition

The nonlinear and nonlocal Hartree equation:

$$\mathrm{i}\partial_t\psi(x,t) = \Big[-\Delta + v(x) + \int \mathrm{d}y\ V(x-y)\,|\psi(y,t)|^2\Big]\psi(x,t)$$

The ingredients are the following:

- $\psi(x,t)$ is the wave function.
- $v(x)$ is the external potential.
- $V(x)$ is the interaction potential (two-body potential).

In which context does this equation appear?

1.1 Bose-Einstein condensation (BEC)

Many body quantum systems: Effective regimes

The Hamiltonian for N (spinless) bosons reads

$$H_N = \sum_{i=1}^{N} (-\Delta_i) + \sum_{i<j} V_N(x_i - x_j).$$

The following scalings are being studied:

- Mean field scaling:
  $V_N(x) = N^{-1} V(x)$
- Gross-Pitaevskii scaling:
  $V_N(x) = N^2 V(Nx)$

Remark Frequent and weak vs. rare and strong

In the limit $N \to \infty$, the many body quantum system can be described by the following effective regimes.

BEC

Mean field regime

$\Longrightarrow$ Hartree equation

The solution of the time dependent Schrödinger equation converges to the solution of the Hartree equation, i.e., for $N \to \infty$, we have, in trace norm,

$$\mathrm{tr}_{N-1} P\left[\mathrm{e}^{-\mathrm{i}tH_N}\psi^{\otimes N}\right] \to P[\psi_t].$$

[Spohn 80], [Erdős-Yau 01] (BBGKY[a]) ; [Hepp 74], [Ginibre-Velo 79], [Pickl 09]

[a] Bogoliubov-Born-Green-Kirkwood-Yvon

Gross-Pitaevskii (GP) regime

$\Longrightarrow$ GP equation

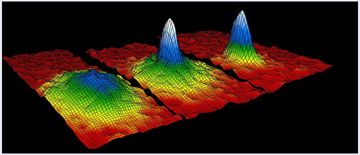
$$\mathrm{i}\partial_t\psi(x,t) = (-\Delta + v(x) + |\psi(x,t)|^2)\psi(x,t)$$

1. The ground state of $H_N + v_N$, where v_N is confining, converges to the GP ground state. [Lieb *et al.* 00, 02]
2. The solution of the time dependent Schrödinger equation converges to the solution of the GP equation. [Erdős *et al.* 07,08]

BEC

Experiments: 1995

1. A gas of $\sim 10^3 - 10^6$ rubidium ${}^{87}_{37}\mathrm{Rb}$, sodium ${}^{23}_{11}\mathrm{Na}$, or lithium ${}^{7}_{3}\mathrm{Li}$ atoms is initially confined in a magnetic trap of size $\sim 10^{-4}m$ and cooled down to $\sim 10^{-6}K$.
2. By optical methods, a sharp peak is recorded in the velocity distribution of the atoms released to free expansion after switching off the trap.
   [Anderson *et al.* 95], [Davies *et al.* 95], [Bradley *et al.* 95]



[Anderson *et al.* 95]
L: temperature just above condensation
M: just after the appearance of the condensate
R: evaporation leaves a (nearly) pure condensate

$\Longrightarrow$ The minimizer of the GP functional describes the condensate in the trap.
$\Longrightarrow$ The time dependent GP equation describes the condensate after switching off the trap.

BEC: Extended scenarios

High density situations

1. Original BEC scenario: the gas is dilute and has repulsive interactions.
2. For attractive interactions, the system collapses as soon as the number of condensate atoms exceeds a critical value $\sim 10^3$. [Bradley *et al.* 95, 97]
3. The collapse has its mathematical manifestation in the nonexistence of a minimizer of the GP functional,

$$\mathcal{E}_{\mathrm{GP}}[\psi] := \mathrm{T}[\psi] - g\mathrm{GP}[\psi],$$

where $g > 0$, and we define the kinetic and the GP energy by

$$\mathrm{T}[\psi] := \tfrac{1}{2}\|\nabla\psi\|^2,$$
$$\mathrm{GP}[\psi] := \|\psi\|_4^4.$$

Namely, using the scaling $\psi_\lambda(x) := \lambda^{3/2}\psi(\lambda x)$, we have, for $\lambda \to \infty$,

$$\mathcal{E}_{\mathrm{GP}}[\psi_\lambda] = \lambda^2\mathrm{T}[\psi] - g\lambda^3\mathrm{GP}[\psi] \to -\infty.$$

BEC: Extended scenarios

4. Even close to collapse, the condensate is significantly affected by mechanisms *not* included in GP theory: important are
   - the internal structure of the bosons,
   - ionization and recombination processes,
   - the interaction with the electromagnetic field.
5. Replacing the GP energy by the Hartree energy,

$$\mathrm{H}[\psi] := (\psi, V * |\psi|^2 \psi),$$

   amounts to replacing the GP scaling by the mean field scaling, and:
   - It accounts for a less coarse-grained resolution of the boson-boson interaction.
   - The existence of a minimizer is always assured (even without a trap).
   - It continues to be mathematically meaningful as the collapse point is approached (and even beyond).
   - It may be expected to give a qualitative account of the *onset* of the collapse.
6. These considerations also apply to the case of repulsive interaction if the density becomes high.

BEC: Extended scenarios

Interacting condensates

1. It is experimentally possible to create *several* different condensates in the same trap repelling each other. [Myatt *et al.* 96], [Hall *et al.* 98], [Stenger *et al.* 98]
2. Two interacting condensates, possibly made of different atomic species (e.g. ${}^{87}_{37}\mathrm{Rb}$ and ${}^{39}_{19}\mathrm{K}$), are described by

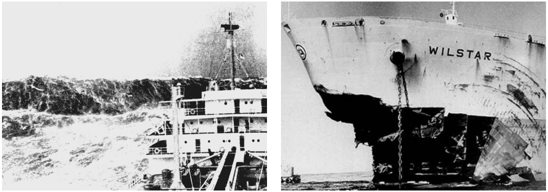
$$\begin{cases} \mathrm{i}\partial_t\psi_1(x,t) = \left(-\Delta + v_1(x) + g_1|\psi_1(x,t)|^2 + \kappa|\psi_2(x,t)|^2\right)\psi_1(x,t) \\ \mathrm{i}\partial_t\psi_2(x,t) = \left(-\Delta + v_2(x) + g_2|\psi_2(x,t)|^2 + \kappa|\psi_1(x,t)|^2\right)\psi_2(x,t) \end{cases}$$

3. For the same reasons as described in the single condensate case, we may replace the local GP energy by the nonlocal Hartree energy,

$$|\psi_i|^2\psi_j \mapsto V * |\psi_i|^2\psi_j .$$

1.2 Monster waves

Observations

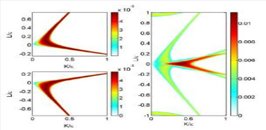


- The estimated loss of carriers of length $> 200m$ is around $10/$year. A prominent case is the complete disappearance of the LASH-carrier *München* in 1978 $(261m)$. [Schmitz-Eggen 02]
- Reported waves suddenly reach heights $\sim 30m$.
- The first measurement was done on the Draupner-E platform in 1995.
- EU project *MaxWave* involved the satellites ERS-1/2: such waves were discovered to be frequent!

Monster waves

Theoretical approaches

1. Different models have been proposed. The linear models all suffered from one or several of the following problems:
   - The predicted frequencies are far too small compared with the satellite data.
   - The speed of the build-up and the reachable height are not big enough.
   - The stability is very short-lived (dispersion).
2. Due to many cross-sea accidents, approximate models, derived from the Navier-Stokes equation, lead to coupled nonlinear Schrödinger equations for the wave envelopes. [Onorato *et al.* 06], [Shukla *et al.* 06]
3. The theoretical and numerical predictions based on this model are in good agreement with the observed data.



4. Derivation from Euler equation using multiscale analysis. [A-Giannoulis]

1.3 Further applications

Miscellanea

1. Newtonian point particle limit
For shallow external potentials $v(\varepsilon x)$, and for superpositions of minimizers

$$\psi(x,t) := \sum_i \Phi_{N_i(t)}(x - r_i(t))\mathrm{e}^{\mathrm{i}\theta_i(x,t)} + o(\varepsilon),$$

the Euler-Lagrange equation w.r.t. r_i of the corresponding action functional leads to Newtons equation of motion,

$$\ddot{r}_i(t) = -\varepsilon\nabla v(\varepsilon r_i) + \frac{\varepsilon}{2}\sum_{j\neq i} N_j\,\nabla V_{\mathrm{lr}}(\varepsilon(r_i(t) - r_j(t)) + o(\varepsilon).$$

It describes the motion of extended particle in shallow external potential weakly interacting with a dispersive environment ($\Rightarrow$ Newtonian gravity).

2. Nonlinear optics
3. Plasma physics
4. Material sciences ...

Large coupling phenomena

We will discuss two *time independent* large coupling phenomena.

Symmetry breaking [A *et al.* 02]

- Consider the Hartree equation with an attractive Coulomb two-body interaction, and let the external potential have a given symmetry.

Then, any ground state exhibits spontaneous symmetry breaking in the large coupling regime.

Phase segregation [A-Squassina 09]

- Consider a system of two coupled Hartree equations with repulsive Coulomb two-body interactions, and let the external potentials be confining.

Then, any system ground state undergoes phase segregation in the large interspecies coupling regime.

2.1 Symmetry breaking: Functional setting

Definition: The function space

$$H^1(\mathbb{R}^3) = \{\psi \in L^2(\mathbb{R}^3) \mid \|\nabla\psi\| < \infty\}$$

Definition: The Hartree functional

Let $v \in C_b(\mathbb{R}^3, \mathbb{R})$, set $V(x) := |x|^{-1}$, and let $g > 0$. The functionals $\mathcal{E}_g, \mathcal{E}_{v,g} : H^1(\mathbb{R}^3) \to \mathbb{R}$ are defined by

$$\mathcal{E}_g[\psi] := \mathrm{T}[\psi] - g\mathrm{H}[\psi],$$
$$\mathcal{E}_{v,g}[\psi] := \mathcal{E}_g[\psi] + \mathrm{v}[\psi],$$

where the kinetic, the external, and the Hartree energy are defined by

$$\mathrm{T}[\psi] := \tfrac{1}{2}\|\nabla\psi\|^2,$$
$$\mathrm{v}[\psi] := (\psi, v\psi),$$
$$\mathrm{H}[\psi] := (\psi, V * |\psi|^2\psi).$$

Remark $|\mathrm{H}[\psi]| \leq C\|\psi\|^3\|\nabla\psi\|$ by Hardy-Littlewood-Sobolev, Gagliardo-Nirenberg.

2.2 Existence theory

Proposition: Existence of a minimizer

Let the external potential $v : \mathbb{R}^3 \to \mathbb{R}$ satisfy one of the following assumptions.

1. It vanishes.
2. It is a localized trap, i.e. $v \in L^{3/2}(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ satisfies $v \leq 0$ with $v \neq 0$, and $v(x) \to 0$ for $|x| \to \infty$.
3. It is a confining trap, i.e. $v \in L^1_{\text{loc}}(\mathbb{R}^3)$ satisfies $v \geq 0$ and $v(x) \to \infty$ for $|x| \to \infty$.

Then, there exists a Hartree minimizer.

Proof. The direct method in the calculus of variations leads to the assertion (and e.g. the concentration compactness principle [Lions 84] for (1) and (2)). $\square$

Remarks

- Note that, if $v \in L^p(\mathbb{R}^3) + L^q(\mathbb{R}^3)$ with $\frac{3}{2} \leq p, q < \infty$ satisfies $v \geq 0$ and $v \neq 0$, no minimizer exists. [Lions 84]
- For *short range* two-body potentials V (e.g. of van der Waals type) and no trap, Birman-Schwinger theory implies the existence of a minimizer for sufficiently large couplings only (e.g. CLR bounds).

2.3 Theorem

Theorem: Symmetry breaking [A *et al.* 02]

Assume that

- $v \in C_b(\mathbb{R}^3, \mathbb{R})$,
- $G \in \mathrm{E}(3)$ is s.t. any minimizing sequence $x_n \in \mathbb{R}^3$ of v satisfies the gap condition

$$\liminf_{n \to \infty} |G(x_n) - x_n| > 0.$$

Then, for sufficiently large $g > 0$, any minimizer Φ of the Hartree functional exhibits spontaneous symmetry breaking,

$$|\Phi \circ G|^2 \neq |\Phi|^2.$$

Remarks

- If the infimum of v is attained on its set M_v of minima, then the gap condition reads $\inf_{x \in M_v} |G(x) - x| > 0$.
- This theorem implies the symmetry breaking formulated earlier: if $v \circ G = v$ for some G satisfying the gap condition, then the minimizer breaks the symmetry of v.

Examples and proof strategy

Examples: External potential

1. Double well
2. Mexican hat
3. Periodic potential
4. For a single well, the gap condition is not satisfied for any $G \in \mathrm{E}(3)$ which is a symmetry of v.

The proof will be carried out in *three steps*:

1. Any approximate free minimizer is arbitrarily strongly concentrated.
2. Any minimizer is an approximate free minimizer, and it is localized around the minima of the external potential.
3. Any invariant minimizer violates mass conservation.

Step1: Any approximate free minimizer is arbitrarily strongly concentrated.

Definition: Mass, free ground state energy, and approximate free minimizers

For any subset $\Omega \subseteq \mathbb{R}^3$ and any $\psi \in L^2(\mathbb{R}^3)$, we define

$$\mathrm{N}_\Omega[\psi] := \|\psi\|^2_{L^2(\Omega)},$$

and $\mathrm{N}[\psi] := \mathrm{N}_{\mathbb{R}^3}[\psi]$. For $0 < \eta < 1$ and $g > 0$, the free ground state energy and the set of approximate free minimizers are defined by

$$E_g := \inf\{\mathcal{E}_g[\psi] \,|\, \psi \in H^1(\mathbb{R}^3) \text{ and } \mathrm{N}[\psi] = 1\},$$
$$\mathcal{M}_g^{(\eta)} := \{\psi \in H^1(\mathbb{R}^3) \,|\, \mathrm{N}[\psi] = 1 \text{ and } \mathcal{E}_g[\psi] \leq (1-\eta)E_g\}.$$

Lemma 1: Concentration

Let $0 < \delta < 1$. Then, there exist $\eta_0 > 0$, $g_0 > 0$, and $y_0 \in \mathbb{R}^3$, s.t., for all $\psi \in \mathcal{M}_g^{(\eta)}$ with $\eta \leq \eta_0$ and $g \geq g_0$, we have

$$\mathrm{N}_{B_\delta(y_0)}[\psi] \geq 1 - \delta.$$

Step1

Proof.

1. We introduce a partition of unity: Let $\chi \in C_0^\infty(\mathbb{R}^3)$ satisfy
   - $\operatorname{supp} \chi \subset B_1(0)$,
   - $0 \leq \chi \leq 1$,
   - $\|\chi\| = 1$.

   Moreover, for any $y \in \mathbb{R}^3$ and any $0 < \delta < 1$, we define
   - $\chi_{y,\delta}(x) := \chi(\frac{x-y}{\delta})$,
   - $j_{y,\delta}(x) := \delta^{-3/2}\chi_{y,\delta}(x)$.

   We then have the following:
   - (Overlap) For any $0 < \gamma < 1$, there exists $\varepsilon > 0$ s.t., for all $x, x' \in \mathbb{R}^3$ with $|x - x'| < \varepsilon$,
   $$\int_{\mathbb{R}^3} \mathrm{d}y \; \chi^2(x-y)\,\chi^2(x'-y) \geq 1 - \gamma.$$
   - (Partition of unity) For any $0 < \delta < 1$, we have, for all $x \in \mathbb{R}^3$,
   $$\int_{\mathbb{R}^3} \mathrm{d}y \; j_{y,\delta}^2(x) = 1.$$
   - (IMS localization) For any $0 < \delta < 1$, we have, with $p := -\mathrm{i}\nabla$,
   $$p^2 = \int_{\mathbb{R}^3} \mathrm{d}y \; j_{y,\delta}\, p^2 j_{y,\delta} - \int_{\mathbb{R}^3} \mathrm{d}y \; (\nabla j_{y,\delta})^2.$$

Step1

Let $0 < \delta < 1$ be fixed.

2. Using IMS localization, we have

$$\mathrm{T}[\psi] = \frac{1}{\delta^3} \int_{\mathbb{R}^3} \mathrm{d}y\ \mathrm{T}[\chi_{y,\delta}\psi] - \frac{1}{\delta^2}\mathrm{T}[\chi].$$

3. Let $0 < \gamma < 1$. Then, using the overlap property, there exists an $\varepsilon > 0$ s.t., integrating over the ball of radius $\varepsilon\delta$ and its complement, we get

$$\mathrm{H}[\psi] \leq \frac{1}{\delta^3}\frac{1}{1-\gamma} \int_{\mathbb{R}^3} \mathrm{d}y\ \mathrm{H}[\chi_{y,\delta}\psi] + \frac{1}{\varepsilon\delta}.$$

4. Using the scaling $S(\alpha, \beta)\psi(x) := \alpha^{3/2}\beta^2\psi(\alpha\beta x)$ with $\alpha, \beta > 0$, we get, for any nonvanishing $\varphi \in H^1(\mathbb{R}^3)$,

$$\mathcal{E}_{\alpha\beta}[\varphi] = \alpha^2\|\varphi\|^6\mathcal{E}_\beta[S(\alpha^{-1}, \|\varphi\|^{-2})\varphi].$$

Step1

5. Plugging (2), (3), and (4) into the functional, we get the lower bound

$$\begin{aligned}\mathcal{E}_g[\psi] &= \mathrm{T}[\psi] - g\mathrm{H}[\psi] \\ &\geq \frac{1}{\delta^3}\int_{\mathbb{R}^3} \mathrm{d}y\ \mathcal{E}_{\frac{g}{1-\gamma}}[\chi_{y,\delta}\psi] - \frac{1}{\delta^2}\mathrm{T}[\chi] - \frac{g}{\varepsilon\delta}\end{aligned}$$

$$= \frac{1}{\delta^3(1-\gamma)^2}\int_{\mathbb{R}^3} \mathrm{d}y\ \|\chi_{y,\delta}\psi\|^6 \underbrace{\mathcal{E}_g[S(1-\gamma, \|\chi_{y,\delta}\psi\|^{-2})\chi_{y,\delta}\psi]}_{\geq E_g = g^2 E_1} - \frac{1}{\delta^2}\mathrm{T}[\chi] - \frac{g}{\varepsilon\delta}.$$

6. Next, we do the following:
   - We use the assumption $\psi \in \mathcal{M}_g^{(\eta)}$, i.e. $\mathcal{E}_g[\psi] \leq (1-\eta)E_g = g^2(1-\eta)E_1$,
   - we divide the inequality by $g^2 E_1 < 0$,
   - we use $\|\chi_{y,\delta}\psi\|^4 \leq \mathrm{N}^2_{B_\delta(y)}[\psi]$.

   Then, we find

$$1-\eta \leq \frac{1}{\delta^3(1-\gamma)^2}\sup_{y\in\mathbb{R}^3} \mathrm{N}^2_{B_\delta(y)}[\psi] \underbrace{\int_{\mathbb{R}^3} \mathrm{d}y\ \|\chi_{y,\delta}\psi\|^2}_{=\delta^3} + \frac{1}{g^2|E_1|}\left(\frac{1}{\delta^2}\mathrm{T}[\chi] + \frac{g}{\varepsilon\delta}\right).$$

This concludes the proof of Lemma 1. □

Step 2: Any minimizer is an approximate free minimizer, and it is localized around the minima of the external potential.

Definition: Ground state energy and minimizers

Let $v \in C_b(\mathbb{R}^3, \mathbb{R})$ and $g > 0$. The ground state energy and the set of minimizers are defined by

$$E_{v,g} := \inf\{\mathcal{E}_{v,g}[\psi] \mid \psi \in H^1(\mathbb{R}^3) \text{ and } \mathrm{N}[\psi] = 1\},$$
$$\mathcal{M}_{v,g} := \{\psi \in H^1(\mathbb{R}^3) \mid \mathrm{N}[\psi] = 1 \text{ and } \mathcal{E}_{v,g}[\psi] = E_{v,g}\}.$$

Lemma 2: Localization

Let $v \in C_b(\mathbb{R}^3, \mathbb{R})$, $0 < \delta < 1$, and $\varepsilon > 0$. Then, there exist $y_* \in \mathbb{R}^3$ and $g_* > 0$ s.t. any $\Phi \in \mathcal{M}_{v,g}$ with $g \geq g_*$,

1. is an approximate free minimizer satisfying

$$\mathrm{N}_{B_\delta(y_*)}[\Phi] \geq 1 - \delta,$$

2. and is localized around the minima of the potential,

$$\inf_{x \in B_\delta(y_*)} v(x) \leq \inf_{x \in \mathbb{R}^3} v(x) + \varepsilon.$$

Step 2

Proof.

1. Pick $\Phi \in \mathcal{M}_{v,g}$ and any $\psi \in H^1(\mathbb{R}^3)$ with $\mathrm{N}[\psi] = 1$. Then, using
   - $0 \geq \mathcal{E}_{v,g}[\Phi] - \mathcal{E}_{v,g}[\psi] = \mathcal{E}_g[\Phi] - \mathcal{E}_g[\psi] + \mathrm{v}[\Phi] - \mathrm{v}[\psi]$,
   - $|\mathrm{v}[\Phi] - \mathrm{v}[\psi]| \leq 2\|v\|_\infty$,

   we can bound the free energy of the minimizer by

$$\mathcal{E}_g[\Phi] \leq E_g\left(1 - \frac{2\|v\|_\infty}{g^2|E_1|}\right).$$

2. To prove the 2nd part, we set $v_* := \inf_{x\in\mathbb{R}^3} v(x)$.
   - We note that, for any $\varepsilon > 0$, there exist $y_1 \in \mathbb{R}^3$ and $\delta_1 > 0$ s.t., for all $x \in B_{\delta_1}(y_1)$, we have $v(x) \leq v_* + \varepsilon/2$ since $v \in C(\mathbb{R}^3, \mathbb{R})$.
   - Moreover, we assume that

$$\inf_{x\in B_\delta(y_*)} v(x) > v_* + \varepsilon.$$

Step 2

3. We define the translated minimizer $\Phi'(x) := \Phi(x - (y_1 - y_*))$, and we show that Φ' has strictly lower energy than Φ. Since
   - $\mathrm{N}_{B_\delta(y_1)}[\Phi'] = \mathrm{N}_{B_\delta(y_*)}[\Phi] \geq 1 - \delta$ and $\mathrm{N}_{\mathbb{R}^3 \setminus B_\delta(y_1)}[\Phi'] \leq \delta$,

   we have, on one hand,

$$\mathrm{v}[\Phi'] - v_* \leq \frac{\varepsilon}{2} + \|v - v_*\|_\infty \underbrace{\mathrm{N}_{\mathbb{R}^3 \setminus B_{\delta_1}(y_1)}[\Phi']}_{\leq \delta, \text{ if } \delta = \mathcal{O}(\varepsilon) \leq \delta_1} .$$

   On the other hand, using the assumption, we get

$$\mathrm{v}[\Phi] - v_* > \varepsilon \underbrace{\mathrm{N}_{B_\delta(y_*)}[\Phi]}_{\geq 1 - \delta, \text{ by 1st part}} - \|v - v_*\|_\infty \underbrace{\mathrm{N}_{\mathbb{R}^3 \setminus B_\delta(y_*)}[\Phi]}_{\leq \delta, \text{ by 1st part}} .$$

4. Hence, plugging these bounds into the energy difference, we get

$$\begin{aligned} \mathcal{E}_{v,g}[\Phi'] - \mathcal{E}_{v,g}[\Phi] &= \mathrm{v}[\Phi'] - v_* - (\mathrm{v}[\Phi] - v_*) \\ &< -\left(\frac{\varepsilon}{2} - \delta(\varepsilon + 2\|v - v_*\|_\infty)\right), \end{aligned}$$

   and the r.h.s. is strictly negative if δ is sufficiently small.

This concludes the proof of Lemma 2. $\square$

Step 3: Any invariant minimizer violates mass conservation.

Proof of the Theorem.

1. The gap condition implies that, for $\varepsilon > 0$ sufficiently small, there exists $0 < \delta < 1/2$ s.t., for any $x \in \mathbb{R}^3$ satisfying $v(x) \leq v_* + 2\varepsilon$, it holds $|G(x) - x| > 4\delta$.
2. It follows from the 2nd part of Lemma 2 that, for any $\varepsilon, \delta > 0$, there exists $x_0 \in B_\delta(y_*)$ s.t. $v(x_0) \leq v_* + 2\varepsilon$. Hence, we have

$$|G(y_*) - y_*| \geq \underbrace{|G(x_0) - x_0|}_{> 4\delta, \text{ by (1)}} - \underbrace{|G(y_*) - G(x_0) + y_* - x_0|}_{\leq 2|x_0 - y_*| \leq 2\delta} > 2\delta$$

3. Let $\Phi \in \mathcal{M}_{v,g}$ with $g \geq g_*$, and assume that $\Phi \circ G = \Phi$. Then, since $\mathrm{N}_{B_\delta(G(y_*))}[\Phi] = \mathrm{N}_{B_\delta(y_*)}[\Phi] \geq 1 - \delta$ due to the 1st part of Lemma 2, we get a contradiction,

$$\begin{aligned} \mathrm{N}[\Phi] &\geq \mathrm{N}_{B_\delta(y_*)}[\Phi] + \mathrm{N}_{B_\delta(G(y_*))}[\Phi] \\ &\geq 2(1-\delta) > 1. \end{aligned}$$

This coludes the proof of the Theorem. □

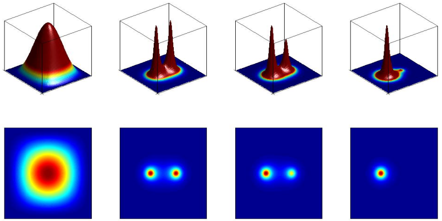
2.4 Remarks

Theorem: Uniqueness [A *et al.* 02]

- The minimizer is unique for sufficiently small coupling.
- Symmetry breaking occurs above a strictly positive critical coupling only.

Size of critical coupling: Numerics [A *et al.* 02]

- v is a double well composed of $(\cosh|x|)^{-1}$-type wells.
- Numerical analysis (FE) of nonlinear iteration procedures [A 02,09].



3. Phase segregation

Definition: The function space

Let $v_1, v_2 \in C(\mathbb{R}^3, \mathbb{R}_0^+)$ be confining, and define

$$\mathcal{H} := \{[\psi_1, \psi_2] \in H^1(\mathbb{R}^3)^{\times 2} \,|\, (\psi_i, v_i\psi_i) < \infty,\ i = 1, 2\}.$$

Definition: The Hartree functional

For $i = 1, 2$, let $g_i > 0$ and $\kappa > 0$, and define the Hartree system functional $\mathcal{E}_\kappa : \mathcal{H} \to \mathbb{R}$ by

$$\mathcal{E}_\kappa[\psi_1, \psi_2] := \sum_{i=1}^{2} \left(\mathrm{T}[\psi_i] + \mathrm{v}_i[\psi_i] + g_i \mathrm{H}[\psi_i]\right) + \kappa \mathrm{H}[\psi_1, \psi_2],$$

where the interspecies Hartree energy (direct term) is given by

$$\mathrm{H}[\psi_1, \psi_2] := (\psi_1, V * |\psi_2|^2 \psi_1).$$

Phase segregation

Theorem: Phase segregation [A-Squassina 09]

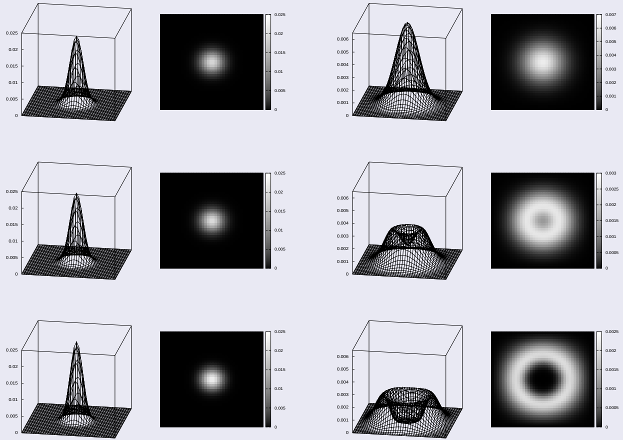
1. For any $\kappa > 0$, there exists a system minimizer $[\Phi_1^\kappa, \Phi_2^\kappa]$.
2. Any sequence of system minimizers features phase segregation in the large interspecies coupling limit, i.e. for $\kappa \to \infty$, we have

$$\mathrm{H}[\Phi_1^\kappa, \Phi_2^\kappa] = o(\kappa^{-1}).$$

Remark Different types of spatial separation exhibited by BE condensates are distinguished by their dynamical behavior after (adiabatically) switching off the confining potentials [Timmermans 98]:

- Potential separation: diffusion into each other
- Phase separation: persists in the absence of external potentials

Spatial separation



4. Outlook

Directions (proposed or in progress)

1. (BEC) Dynamical potential/phase separation.
2. (BEC) More general interacting systems (BEC triplet states).
3. (Magnetic fields) Existence of system minimizers.
4. (Monster waves) Derivation of the system of coupled nonlinear Schrödinger equations from Navier-Stokes equation. [A-Giannoulis]
5. (Newtonian limit) Soliton dissipation phenomena: return to equilibrium [A], coalescence, structure formation, ...
6. Many more ...

Thank you for your attention!

Katsushika Hokusai *The great wave off Kanagawa* (woodcut, ~1830)