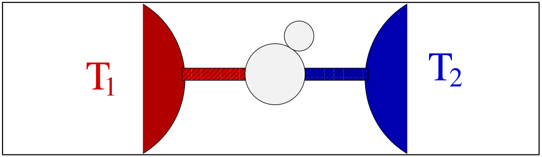
Broken translation invariance in quasifree fermionic correlations out of equilibrium



Walter H. Aschbacher (Ecole Polytechnique)

CPT Marseille April 2011

What's the physical situation we are interested in?

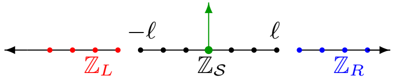
Question:

- Open systems

Fundamental paradigm

A confined sample is suitably coupled to two extended thermal reservoirs at different temperatures s.t., for large times, the system approaches a unique nonequilibrium steady state (NESS).

- We consider quasifree fermionic systems over the discrete line whose translation invariance has been broken by a local magnetization $\kappa > 0$:



- We ask: *What is the mathematical and physical effect of the breaking of translation invariance on the NESS expectation value of (an important class of) spatial correlations?*

What's the physical situation we are interested in?

Quasifree fermions play an important role in the study of open systems:

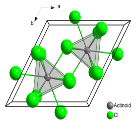
- They allow for a powerful description by means of scattering theory on the one-particle Hilbert space of the observable algebra.
- They are realized in nature.

Specific instance: XY chain [Lieb *et al.* 61, Araki 84]

- The Heisenberg Hamiltonian density reads (XY chain: $J_3 = 0$)

$$H_x = \sum_{i=1,2,3} J_i \sigma_i^x \sigma_i^{x+1} + \lambda \sigma_3^x .$$

Experiments $SrCuO_2$, Sr_2CuO_3 [Sologubenko *et al.* 01] with $J_3 \neq 0$
$PrCl_3$ [Culvahouse *et al.* 69, D'Iorio *et al.* 83] with $J_1 = J_2,\ J_3 \approx 0,\ \lambda = 0$



What's the physical situation we are interested in?

Correlation observable

- The correlation observable whose NESS expectation we study is the ferromagnetic string of length n starting at site x_0 ,

Emptiness Formation Probability (EFP)

$$\prod_{x=x_0}^{x_0+n-1} \frac{1-\sigma_3^{(x)}}{2},$$

introduced in [Korepin *et al.* 94] (*pour fixer les idées*).

- After a Araki-Jordan-Wigner transformation [Jordan-Wigner 28, Araki 84] mapping the spin system onto free fermions, the EFP becomes

$$\prod_{x=x_0}^{x_0+n-1} a_x a_x^*.$$

Description

Physical/Mathematical ingredients

1. Quantum statistical mechanics (Operator algebra approach)
2. NESS (Hilbert space scattering theory)
3. Correlation asymptotics (Functional analysis of Toeplitz operators)

References

- *Broken translation invariance in quasifree fermionic correlations out of equilibrium*
  J. Funct. Anal. 260 (2011) 3429–56 (arXiv:1103.4512)
- *A remark on the subleading order in the asymptotics of the nonequilibrium emptiness formation probability*
  Confluentes Math. 2 (2010) 293–311 (arXiv:1009.1584)
- *On the emptiness formation probability in quasi-free states*
  Contemp. Math. 447 (2007) 1–16 (mp_arc 07-34)

Reminder: Algebraic formalism of quantum statistical mechanics

Rigorous foundation in the early 1930s:

- An observable A is a selfadjoint operator on the Hilbert space of the system.
- The dynamics of the system is determined by a distinguished selfadjoint operator H, called the Hamiltonian, through $A \mapsto A_t = \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH}$.
- A pure state is a vector ψ in the Hilbert space, and the expectation value of the measurement of A in the state ψ is $(\psi, A\psi)$.

Algebraic reformulation and generalization (von Neumann, Jordan, Wigner, ...):

Observables

C^* algebra $\mathfrak{A}$

Dynamics

(Strongly) continuous group τ^t of *-automorphisms on $\mathfrak{A}$

States

Normalized positive linear functionals ω on $\mathfrak{A}$, denoted by $\mathcal{E}(\mathfrak{A})$

Example $\mathfrak{A} = \mathcal{L}(\mathfrak{h})$, $\tau^t(A) = \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH}$, and $\omega(A) = \mathrm{tr}(\varrho A)$

1.1 General setting

Observables

- The algebra of observables has the following structure:

Selfdual CAR [Araki 71]

Let $\mathfrak{H}$ be a complex Hilbert space and J an antiunitary involution. A *self-dual CAR algebra* $\mathfrak{A}(\mathfrak{H}, J)$ is the C^* completion of the $*$ algebra generated by $B(F)$, $B^*(F)$ for $F \in \mathfrak{H}$, and an identity 1 s.t.

1. $B(F)$ is complex linear in F,
2. $\{B^*(F), B(G)\} = (F, G)1$,
3. $B^*(F) = B(JF)$.

- Projection $P \in \mathcal{L}(\mathfrak{H})$ satisfying $JPJ = 1 - P$ (*basis projection*): selfdual CAR $*$-isomorphic to usual CAR over $\mathrm{ran}(P)$.
- This is the natural framework to treat quasifree fermionic systems.

1.1 General setting

Dynamics

- The time evolution has the following structure:

Bogoliubov $*$-automorphisms [Araki 71]

A *Bogoliubov transformation* is a unitary operator $U \in \mathcal{L}(\mathfrak{H})$ satisfying $[J, U] = 0$ which defines a *Bogoliubov $*$-automorphism* by

$$\tau_U(B(F)) := B(UF).$$

- For the special case of the unitary group $U_t = \mathrm{e}^{-\mathrm{i}tH}$, where $H \in \mathcal{L}(\mathfrak{H})$ is self-adjoint and $\{H, J\} = 0$, we set

$$\tau^t(B(F)) := B(U_{-t}F).$$

1.1 General setting

States

- States are normalized positive linear functionals ω on $\mathfrak{A}$, denoted $\mathcal{E}(\mathfrak{A})$.
- For the nonequilibrium situation, we use:

NESS [Ruelle 01]

A *NESS* $\omega \in \mathcal{E}(\mathfrak{A})$ associated with the C^*-dynamical system $(\mathfrak{A}, \tau)$ and the initial state $\omega_0 \in \mathcal{E}(\mathfrak{A})$ is a weak-$*$ limit point for $T \to \infty$ of

$$\left\{ \frac{1}{T} \int_0^T \mathrm{d}t \; \omega_0 \circ \tau^t \,\middle|\, T > 0 \right\}.$$

- The two-point function is characterized as follows:

Density

The *density* of a state $\omega \in \mathcal{E}(\mathfrak{A})$ is the operator $S \in \mathcal{L}(\mathfrak{H})$ satisfying $0 \leq S^* = S \leq 1$ and $JSJ = 1 - S$, and, for all $F, G \in \mathfrak{H}$,

$$\omega(B^*(F)B(G)) = (F, SG).$$

1.1 General setting

States

- The class of states we are concerned with:

Quasifree states [Araki 71]

A state $\omega \in \mathcal{E}(\mathfrak{A})$ is called *quasifree* if it vanishes on odd polynomials in the generators, and if it is a Pfaffian on the even polynomials, *i.e.*

$$\omega(B(F_1)...B(F_{2n})) = \mathrm{pf}(\Omega_n),$$

where $\Omega_n \in \mathbb{C}^{2n\times 2n}$ is defined to be skew-symmetric and, for $i < j$,

$$\Omega_{ij} := \omega(B(F_i)B(F_j))$$

The *Pfaffian* $\mathrm{pf} : \mathbb{C}^{2n\times 2n} \to \mathbb{C}$ is defined on all skew-symmetric matrices A by

$$\mathrm{pf}(A) := \sum_{\pi} \mathrm{sign}(\pi) \prod_{j=1}^{n} A_{\pi(2j-1),\pi(2j)},$$

where the sum is running over all pairings π of $\{1, 2, ..., 2n\}$.

1.2 Specific model

Observables

- Selfdual CAR $\mathfrak{A}(\mathfrak{H}, J)$:

$\mathfrak{H}$, $B(F)$, and J

The one-particle Hilbert space is $\mathfrak{H} := \mathfrak{h}^{\oplus 2}$, and $\mathfrak{h} := \ell^2(\mathbb{Z})$ reads

$$\mathfrak{h} = \mathfrak{h}_L \oplus \mathfrak{h}_S \oplus \mathfrak{h}_R,$$

where $\mathfrak{h}_\alpha := \ell^2(\mathbb{Z}_\alpha)$. The complex linear mapping $B : \mathfrak{H} \to \mathcal{L}(\mathfrak{F}(\mathfrak{h}))$ is defined, for $F := [f_1, f_2] \in \mathfrak{H}$, by

$$B(F) := a^*(f_1) + a(\bar{f}_2).$$

The antiunitary involution acts as $J[f_1, f_2] := [\bar{f}_2, \bar{f}_1]$.

1.2 Specific model

Observables

- EFP correlation observable and expectation value:

A_n and $\mathrm{P}(n)$

The EFP observable $A_n \in \mathfrak{A}$ reads

$$A_n := \prod_{i=1}^{2n} B(F_i),$$

where, using the translation $u \in \mathcal{L}(\mathfrak{h})$, the *form factors* $F_i \in \mathfrak{H}$ and the *initial form factors* $G_1, G_2 \in \mathfrak{H}$ are given by

$$F_{2i-1} := u^i \oplus u^i G_1, \quad F_{2i} := u^i \oplus u^i G_2, \quad G_1 := JG_2 := [0, \delta_{x_0-1}].$$

The expectation value $\mathrm{P} : \mathbb{N} \to [0,1]$ of $A_n \in \mathfrak{A}$ in the NESS $\omega_{\mathrm{B}} \in \mathcal{E}(\mathfrak{A})$ to be constructed below, is denoted by

$$\mathrm{P}(n) := \omega_{\mathrm{B}}(A_n).$$

1.2 Specific model

Dynamics

- Bogoliubov automorphisms:

XY, decoupled, and magnetic Hamiltonians

The XY and the decoupled Hamiltonian $h, h_0 \in \mathcal{L}(\mathfrak{h})$ are defined by

$$
\begin{aligned}
h &:= \mathrm{Re}(u) && \text{(coupled, translation invariant),}\\
p_0 &:= (\delta_0, \cdot)\delta_0 && \text{(localizes at the origin),}\\
v_L &:= \mathrm{Re}(u^{-(\ell+1)} p_0 u^{\ell}) && \text{(couples left reservoir),}\\
v_R &:= \mathrm{Re}(u^{\ell} p_0 u^{-(\ell+1)}) && \text{(couples right reservoir),}\\
h_0 &:= h - (v_L + v_R) && \\
&= h_L \oplus h_{\mathcal{S}} \oplus h_R && \text{(decouples subsystems).}
\end{aligned}
$$

The magnetic Hamiltonian $h_{\mathrm{B}} \in \mathcal{L}(\mathfrak{h})$ of coupling strength $\kappa > 0$ is

$$
\begin{aligned}
v &:= p_0,\\
h_{\mathrm{B}} &:= h + \kappa v \quad \text{(coupled, broken translation invariance).}
\end{aligned}
$$

- For any operator $a \in \{h, p_0, v_L, v_R, v, h_{\mathrm{B}}\}$, we set $A := a \oplus -a$.

1.2 Specific model

States

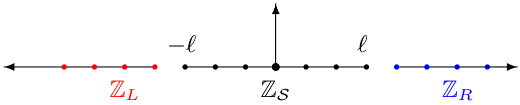
- Initial state for NESS construction:

Decoupled system

Let $0 < \beta_L < \beta_R < \infty$ be the inverse temperatures. The initial state $\omega_0 \in \mathcal{E}(\mathfrak{A})$ is specified by the density $S_0 \in \mathcal{L}(\mathfrak{H})$ given by

$$S_0 := (1 + \mathrm{e}^{-K_0})^{-1},$$

where $K_0 := \beta_L H_L \oplus 0 \oplus \beta_R H_R \in \mathcal{L}(\mathfrak{H})$.



2. Correlation expectation

Proposition: Correlation expectation [A 11]

There exists a unique quasifree NESS $\omega_{\mathrm{B}} \in \mathcal{E}(\mathfrak{A})$ associated with the C^*-dynamical system $(\mathfrak{A}, \tau_{\mathrm{B}})$ and the initial state $\omega_0 \in \mathcal{E}(\mathfrak{A})$, and

$$\mathrm{P}(n) = \mathrm{pf}(\Omega_n^{\mathrm{aa}} + \Omega_n^{\mathrm{pp}}).$$

The *asymptotic correlation matrices* $\Omega_n^{\mathrm{aa}}, \Omega_n^{\mathrm{pp}} \in \mathbb{C}^{2n\times 2n}$ are defined, $i<j$, by

$$\begin{aligned}
\Omega_{ij}^{\mathrm{aa}} &:= \omega_0(B^*(W(H_0,H_{\mathrm{B}})JF_i)B(W(H_0,H_{\mathrm{B}})F_j)),\\
\Omega_{ij}^{\mathrm{pp}} &:= \sum_{e\,\in\,\mathrm{spec}_{\mathrm{pp}}(H_{\mathrm{B}})} \omega_0(B^*(1_e(H_{\mathrm{B}})JF_i)B(1_e(H_{\mathrm{B}})F_j)).
\end{aligned}$$

The wave operator $W(H_0, H_{\mathrm{B}}) \in \mathcal{L}(\mathfrak{H})$ is defined by

$$W(H_0,H_{\mathrm{B}}) := \mathrm{s}-\lim_{t\to\infty}\mathrm{e}^{-\mathrm{i}tH_0}\mathrm{e}^{\mathrm{i}tH_{\mathrm{B}}}1_{\mathrm{ac}}(H_{\mathrm{B}}).$$

2. Correlation expectation

Proof.

1. [*Correlation decomposition*] The NESS expectation has the form

$$\mathrm{P}(n) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{d}t\ \omega_0(\tau_{\mathrm{B}}^t(A_n)) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{d}t\ \mathrm{pf}(\Omega_n(t)),$$

   where τ_{B}^t is the Bogoliubov automorphism generated by H_{B}, and the skew-symmetric $\Omega_n(t) \in \mathbb{C}^{2n\times 2n}$ is defined, for $i < j$, by

$$\Omega_{ij}(t) := \omega_0(B(\mathrm{e}^{\mathrm{i}tH_{\mathrm{B}}}F_i)B(\mathrm{e}^{\mathrm{i}tH_{\mathrm{B}}}F_j)).$$

   Since $\mathrm{spec}_{\mathrm{sc}}(H_{\mathrm{B}}) = \emptyset$ and $\mathrm{spec}_{\mathrm{pp}}(H_{\mathrm{B}}) = \{\pm e_{\mathrm{B}}\}$ [Hume-Robinson 86], we decompose $\Omega_n(t) = \sum_{\alpha,\beta\in\{\mathrm{a},\mathrm{p}\}} \Omega_n^{\alpha\beta}(t)$ $(\mathrm{a}/\mathrm{p} \equiv \mathrm{ac}/\mathrm{pp})$, where

$$\Omega_{ij}^{\alpha\beta}(t) := (\mathrm{e}^{\mathrm{i}tH_{\mathrm{B}}}1_\alpha(H_{\mathrm{B}})JF_i, S_0\mathrm{e}^{\mathrm{i}tH_{\mathrm{B}}}1_\beta(H_{\mathrm{B}})F_j).$$

2. Correlation expectation

2. [*Case* $\alpha, \beta = \mathrm{ac}$] Since $[H_0, S_0] = 0$, we can write

$$\Omega_{ij}^{\mathrm{aa}}(t) = (\mathrm{e}^{-\mathrm{i}tH_0}\mathrm{e}^{\mathrm{i}tH_\mathrm{B}}1_{\mathrm{ac}}(H_\mathrm{B})JF_i, S_0\mathrm{e}^{-\mathrm{i}tH_0}\mathrm{e}^{\mathrm{i}tH_\mathrm{B}}1_{\mathrm{ac}}(H_\mathrm{B})JF_j).$$

Using trace class scattering theory for $t \to \infty$, the wave $W(H_0, H_\mathrm{B})$ exists [Kato-Rosenblum 57] theory, we get Ω_n^{aa}.

3. [*Case* $\alpha \neq \beta$] Since $1_{\mathrm{pp}}(H_\mathrm{B}) \in \mathcal{L}^0(\mathcal{H})$, we have, for $t \to \infty$,

$$|\Omega_{ij}^{\mathrm{ap}}(t)| \leq \| \underbrace{1_{\mathrm{pp}}(H_\mathrm{B})S_0}_{\in \mathcal{L}^\infty(\mathcal{H})} \mathrm{e}^{\mathrm{i}tH_\mathrm{B}}1_{\mathrm{ac}}(H_\mathrm{B})JF_i\| \|F_j\| \longrightarrow 0.$$

4. [*Case* $\alpha, \beta = \mathrm{pp}$] This term has the form

$$\Omega_{ij}^{\mathrm{pp}}(t) = \sum_{e,e' \in \{\pm e_\mathrm{B}\}} \mathrm{e}^{-\mathrm{i}t(e-e')}(1_e(H_\mathrm{B})JF_i, S_0 1_{e'}(H_\mathrm{B})F_j).$$

For $\mathrm{ran}\, 1_{e_\mathrm{B}}(H_\mathrm{B}) \subset \mathfrak{h} \oplus 0$ and $\mathrm{ran}\, 1_{-e_\mathrm{B}}(H_\mathrm{B}) \subset 0 \oplus \mathfrak{h}$ [Hume-Robinson 86] and due to the block diagonal structure of S_0, we get $\Omega_n^{\mathrm{pp}}(t) = \Omega_n^{\mathrm{pp}}$. □

3. Correlation structure

Proposition: Correlation structure [A 11]

The NESS EFP is the determinant of the finite section of a Toeplitz operator, a Hankel operator, and an operator of finite rank. The symbol $a \in L^\infty(\mathbb{T})$ of the Toeplitz operator reads

$$a = \varphi_{\mathrm{B}} s_L + (1 - \varphi_{\mathrm{B}}) s_R,$$

where the functions $\varphi_{\mathrm{B}}, s_\alpha \in L^\infty(\mathbb{T})$ are defined by

$$\begin{aligned} s_\alpha(k) &:= \frac{1}{2}\left(1 - \tanh[\tfrac{1}{2}\beta_\alpha \cos(k)]\right), \\ \varphi_{\mathrm{B}}(k) &:= \chi_{[0,\pi]}(k)\, \frac{\sin^2(k)}{\sin^2(k) + \kappa^2}. \end{aligned}$$

Moreover, the symbol of the Hankel operator is smooth.

3. Correlation structure

Reminder: Toeplitz and Hankel operators

- A theorem by [Toeplitz 11]:
  Let $\{a_x\}_{x\in\mathbb{Z}} \subset \mathbb{C}$. Then, the operator on $\ell^2(\mathbb{N})$ defined through

$$f \mapsto \left\{ \sum_{j=1}^{\infty} a_{i-j} f_j \right\}_{i=1}^{\infty}$$

  is bounded iff there exists a symbol $a \in L^\infty(\mathbb{T})$ s.t.

$$a_x = \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi}\, a(k)\, \mathrm{e}^{-\mathrm{i}kx}.$$

  In this case, we write $T[a] \in \mathcal{L}(\ell^2(\mathbb{N}))$, and we set $T_n[a] := P_n T[a] P_n$, where $P_n\{x_1, x_2, \ldots\} := \{x_1, \ldots, x_n, 0, 0, \ldots\}$.
- A theorem by [Nehari 57]:
  The action is now $f \mapsto \left\{ \sum_{j=1}^{\infty} a_{i+j-1} f_j \right\}_{i=1}^{\infty}$, and we write $H[a]$, $H_n[a]$.

3. Correlation structure

Proof. From the previous proposition, we know that

$$\mathrm{P}(n) = \mathrm{pf}(\Omega_n^{\mathrm{aa}} + \Omega_n^{\mathrm{pp}}).$$

1. [ac-*contribution*] By the chain rule for wave operators, we have

$$\Omega_{ij}^{\mathrm{aa}} = (W(H, H_{\mathrm{B}})JF_i, \underbrace{W(H_0, H)^* S_0 W(H_0, H)}_{= S \in \mathcal{L}(\mathfrak{H})} W(H, H_{\mathrm{B}})F_j),$$

where S is the density of the translation invariant NESS [A-Pillet 03]. Using stationary trace class scattering theory in its weak abelian form,

$$(F, W(H, H_{\mathrm{B}})G) = \int_{-1}^{1} \mathrm{d}e \lim_{\varepsilon \to 0+} \frac{\varepsilon}{\pi}(R_{e \pm \mathrm{i}\varepsilon}(H)F, R_{e \pm \mathrm{i}\varepsilon}(H_{\mathrm{B}})G),$$

expressing $R_{e\pm\mathrm{i}\varepsilon}(H_{\mathrm{B}})$ by H, and switching to the energy space of H (*p.a.c.*), the wave operator can be explicitly determined ($\simeq \delta$-interaction).

3. Correlation structure

2. [pp-*contribution*] We have

$$\Omega^{\mathrm{pp}}_{ij} = \sum_{e \in \{\pm e_{\mathrm{B}}\}} (1_e(H_{\mathrm{B}}) J F_i, S_0 1_e(H_{\mathrm{B}}) F_j),$$

where $S_0 \in \mathcal{L}(\mathfrak{H})$ is the density of the initial state. Using the absence of embedded eigenvalues in $\mathrm{spec}_{\mathrm{ac}}(H_{\mathrm{B}}) = [-1, 1]$ [Hume-Robinson 86], the exponentially localized eigenfunctions and the simple eigenvalues $\pm e_{\mathrm{B}}$ can be explicitly determined.

3. [*Pfaffian reduction*] Orthogonally transforming the correlation matrix into off-diagonal block form and using basic properties of the Pfaffian, we get

$$\mathrm{P}(n) = \det\left(\Omega^{\mathrm{red}}_n\right),$$

where $\Omega^{\mathrm{red}}_n \in \mathbb{C}^{n \times n}$ is given by expressions of the form $\Omega^{\mathrm{aa}}_{2i-1\,2j} + \Omega^{\mathrm{pp}}_{2i-1\,2j}$.

4. [*Toeplitz/Hankel extraction*] Inserting the ac/pp-contributions, we arrive at

$$\mathrm{P}(n) = \det(0 \oplus (T_{n-n_0}[a] + H_{n-n_0}[b]) + M_n),$$

where $M \in \mathcal{L}^0(\mathbb{C}^{n_0} \oplus \ell^2(\mathbb{N}))$ and $b = \mathcal{O}(\kappa) \in L^\infty(\mathbb{T})$ is smooth. □

4. Correlation asymptotics

Theorem: Correlation asymptotics [A 11]

For $n \to \infty$, the NESS EFP has an exponentially decaying bound,

$$\mathrm{P}(n) = \mathcal{O}\big(\mathrm{e}^{-\Gamma n}\big).$$

The decay rate $\Gamma := \Gamma_R + \Gamma_\mathrm{B} > 0$ contains the two parts

$$\begin{aligned}
\Gamma_R &:= -\frac{1}{2}\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\,\log[s_R(k)],\\
\Gamma_\mathrm{B} &:= -\frac{1}{2}\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\,\log[\sigma_\mathrm{B}(k)s_L(k) + (1-\sigma_\mathrm{B}(k))s_R(k)],
\end{aligned}$$

where the function $\sigma_\mathrm{B} \in L^\infty(\mathbb{T})$ is given by

$$\sigma_\mathrm{B}(k) := \frac{\sin^2(k)}{\sin^2(k) + \kappa^2}.$$

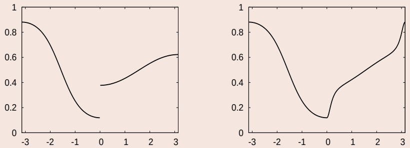
4. Correlation asymptotics

Proof.

1. [*Invertibility and continuity*] Since the symbol $a \in L^\infty(\mathbb{T})$ is real-valued, we use [Hartman-Wintner 54]:
   - If $a \in L^\infty(\mathbb{T})$ is real-valued, then $\mathrm{spec}(T[a]) = \mathrm{conv}(\mathrm{ess}{-}\mathrm{ran}(a))$.

   For $a \in C(\mathbb{T})$, we get $\mathrm{spec}(T[a]) = \frac{1}{2}[1 - \tanh(\frac{1}{2}\beta_R), 1 + \tanh(\frac{1}{2}\beta_R)]$ s.t.

$$0 \notin \mathrm{spec}(T[a]).$$



The symbol $a > 0$ for $\beta_R = 2$, $\beta_L = \frac{1}{2}$, and $\kappa = 0$ (left) and $\kappa = \frac{1}{5}$ (right).

2. [*Stability*] Moreover, using [Gohberg-Feldman 74]:
   - If $a \in C(\mathbb{T})$ and if $T[a]$ is invertible, then $\{T_n[a]\}_{n \in \mathbb{N}}$ is stable.

   we have

$$\limsup_{n \to \infty} \|T_n[a]^{-1}\| < \infty.$$

4. Correlation asymptotics

3. [*Factorization*] In order to use Strong [Szegő 52] for $n \to \infty$ of $\det(T_n[a])/G(a)^n$ with $G(a) := \exp([\log(a)]_0)$ (below), we factorize

$$\frac{\mathrm{P}(n)}{G(a)^n} = \underbrace{\frac{\mathrm{P}(n)}{\det(T_{n-n_0}[a])}}_{\text{Peller \& Separation}} \underbrace{\frac{\det(T_{n-n_0}[a])}{\det(T_n[a])}}_{\text{First Szegő}} \underbrace{\frac{\det(T_n[a])}{G(a)^n}}_{\text{Strong Szegő}},$$

where $n_0 := |x_0|$ if $x_0 < 0$ and zero otherwise.

4. [*First factor*] Due to the determinantal structure, we have

$$\frac{\mathrm{P}(n)}{\det(T_{n-n_0}[a])} = \det(1 + 1 \oplus T^{-1}_{n-n_0}[a]((-1) \oplus H_{n-n_0}[b] + M_n)).$$

Since $b \in C^\infty(\mathbb{T}) \subset L^\infty(\mathbb{T}) \cap B^1_1(\mathbb{T})$, we use [Peller 80]:

- If $b \in L^\infty(\mathbb{T})$, then $H[b]$ is trace class iff $b \in B^1_1(\mathbb{T})$.

and get

$$H[b] \in \mathcal{L}^1(\ell^2(\mathbb{N})).$$

Hence, the first factor converges due to the [Separation Lemma]:

- If A is invertible, $P_n A P_n$ is stable, and K trace class, then $\det(P_n(A+K)P_n)/\det(P_n A P_n) \to \det(1 + A^{-1}K)$.

4. Correlation asymptotics

5. [*Second factor*] Using First [Szegő 15]:
   - If $a \in C(\mathbb{T})$, a real-valued, $\operatorname{ran}(a) \subset (0, \infty)$, and $T[a]$ is invertible, then $\det(T_n[a]) / \det(T_{n-1}[a]) \to G(a)$.

   we have

$$\frac{\det(T_{n-n_0}[a])}{\det(T_n[a])} = \prod_{i=1}^{n_0} \frac{\det(T_{n-i}[a])}{\det(T_{n+1-i}[a])} \to G(a)^{x_0}.$$

6. [*Third factor*] In order to apply Strong [Szegő 52]:
   - If $a \in W(\mathbb{T}) \cap B_2^{1/2}(\mathbb{T})$ has no zeroes on $\mathbb{T}$ and $\operatorname{ind}(a) = 0$, then $\det(T_n[a])/G(a)^n$ converges, where $G(a) = \exp([\log(a)]_0)$.

   we show that

$$\begin{aligned} a &\in C^1(\mathbb{T}) \cap PC^\infty(\mathbb{T}) \subset W(\mathbb{T}) \cap B_2^{1/2}(\mathbb{T}), \\ a &> 0, \\ \operatorname{ind}(a) &= 0. \end{aligned}$$

7. [*Decay rate*] From the 0th Fourier coefficient $[\log(a)]_0$, we arrive at the decay rate $\Gamma := \log G(a)$ of the bound on the exponential decay. □

5. Remarks

Decay rates

$$\Gamma_\alpha = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}k}{2\pi} \log\left[\tfrac{1}{4}(1-\tau_\alpha^2)\right]$$

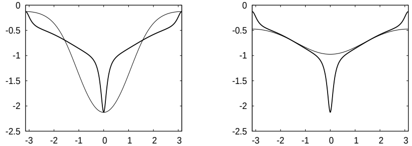
$$\Gamma_\mathrm{B} = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}k}{2\pi} \log\left[\tfrac{1}{4}(1-[\sigma_\mathrm{B}\tau_L + (1-\sigma_\mathrm{B})\tau_R]^2)\right]$$

Here: $\tau_\alpha(k) = \tanh[\frac{1}{2}\beta_\alpha \cos(k)])$ and $\sigma_\mathrm{B}(k) = \sin^2(k)/(\sin^2(k) + \kappa^2)$

1. **(Small coupling)** The theorem holds for any impurity strength κ. In the limit of small κ, we recover the exact decay rate from [A 10] and the bound for general quasifree systems from [A 07].
2. **(Left mover-right mover)** Left movers carrying temperature T_R, left movers reflected at the impurity, and transmitted right movers carrying T_L (see also [A-Barbaroux 07], [A 07, 10] for different NESS correlators).

5. Remarks

3. **(Ordering)** If the system is truly out of equilibrium, we have $\Gamma_R > \Gamma_{\rm B} > \Gamma_L > 0$ (as opposed to the full XY phase diagram).



For $\beta_R = 2$ and $\beta_L = \frac{1}{2}$, the integrand of Γ_R (left, thin line), Γ_L (right, thin line), and of $\Gamma_{\rm B}$ with $\kappa = \frac{1}{5}$ (thick line).

4. **(Regularization)** $\kappa > 0$ regularizes the underlying Toeplitz theory in the sense that the symbol characterizing the decay rate is smoother than in the case $\kappa = 0$. The latter case requires Fisher-Hartwig theory [Ehrhardt-Silbermann 96] and leads to a strictly positive power law subleading order [A 10].

5. Remarks

5. The study for the *anisotropic* XY model is a priori more complicated:

   (Quasifree structure) Due to the ergodic mean, the Pfaffian form of the correlation cannot be preserved in the present form.

   (Toeplitz theory) The symbol of the Toeplitz operator involved becomes *nonscalar* and *nonregular*. But, for block symbols, it is in general difficult to establish invertibility since [Coburn 66]'s Lemma does not hold. And, out of equilibrium, the regularity is lost.

6. **(Different approaches)** It can be advantageous to recast the correlation into the form of a Fredholm determinant, maybe with the help of the [Borodin-Okounkov 00] formula (regularity...).
7. Interesting interplay between physics out of equilibrium, operator algebras, and asymptotic analysis: *To be continued!*